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Asymptotic expansions for laminar forced-convection heat and mass transfer

Part 2. Boundary-layer flows

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This is the second of two articles by the authors dealing with asymptotic expansions for forced-convection heat or mass transfer to laminar flows. It is shown here how the method of the first paper (Acrivos & Goddard 1965), which was used to derive a higher-order term in the large Péclet number expansion for heat or mass transfer to small Reynolds number flows, can yield equally well higher-order terms in both the large and the small Prandtl number expansions for heat transfer to laminar boundary-layer flows. By means of this method an exact expression for the first-order correction to Lighthill's (1950) asymptotic formula for heat transfer at large Prandtl numbers, as well as an additional higher-order term for the small Prandtl number expansion of Morgan, Pipkin & Warner (1958), are derived. The results thus obtained are applicable to systems with non-isothermal surfaces and arbitrary planar or axisymmetric flow geometries. For the latter geometries a derivation is given of a higher-order term in the Péclet number expansion which arises from the curvature of the thermal layer for small Prandtl numbers. Finally, some applications of the results to 'similarity' flows are also presented.

1. Introduction

In an earlier paper (Acrivos & Goddard 1965) we reviewed briefly the general problem of obtaining a solution to the laminar forced-convection heat transfer equation for large Péclet numbers Pe. Then, for the special case of laminar flows with small or moderate Reynolds numbers Re, we derived a general formula for the first 'correction' term, in Pe, to the well-known asymptotic ($Pe \to \infty$) expression for steady-state heat transfer from non-isothermal surfaces, which was applicable to quite general, planar or axisymmetric, flows.

In the present paper, we wish to consider the somewhat related problem of obtaining higher-order approximations to the asymptotic rate of heat transfer for laminar 'boundary-layer' flows, in the limits of large or small Prandtl numbers Pr. In other words, whereas our earlier article (Acrivos & Goddard 1965) dealt with the forced-convection problem for which $Pe \equiv (PrRe) \to \infty$ *with Re fixed*, this second paper will be concerned with the limiting subcases

$Pr \to \infty$ and $Pr \to 0$ of the well-known 'boundary-layer' problem, involving now both thermal as well as viscous boundary layers, for which $PrRe \to \infty$ *with Pr fixed.* Thus, the present work represents a continuation and extension of the previous analyses of Morgan *et al.* (1958), Merk (1959) and Meksyn (1961).

We recall at this point that Merk (1959) and Meksyn (1961) have already derived several higher-order terms in the asymptotic expansion of the Nusselt number for large Pr, but their results are restricted to the rather special case of boundary layers with 'similar' velocity and temperature profiles. In contrast, while we shall obtain here, in § 3, only one 'correction' term to the asymptotic ($Pr \to \infty$) expression for heat transfer (Lighthill 1950; Levich 1962), our result will be applicable to quite general, planar or axisymmetric, boundary layers past non-isothermal surfaces.

As for the other limiting case, that of small Pr, the previous work of Morgan *et al.* (1958) has already yielded, for planar flows, one correction term to the classical Boussinesq (1905) asymptotic expression for heat transfer with $Pr \to 0$. The purpose of the analysis to be presented in § 4 will be then to derive an additional higher-order term in Pr as well as a term in Pe, the latter arising from the curvature of axisymmetric thermal layers.

2. Basic equations

The problem at hand is a special case of the more general one of determining the steady-state rate of heat transfer from a given surface, on which the temperature has prescribed values, to an adjacent fluid stream in laminar boundary-layer flow. In particular, our analysis will be restricted further to incompressible, 'constant-property' flows and to systems with either planar or axisymmetric velocity and temperature fields for which in principle the velocity profile can be specified *a priori* in terms of a stream function $U_\infty L\psi(x,y)$, where x and y, the independent variables, are the usual dimensionless 'boundary-layer' co-ordinates, with xL and yL measuring, respectively, the distance along and normal to the surface (at $y = 0$). (We have chosen the quantities L and U_∞ to represent here a characteristic length and a characteristic velocity for the system; also, we shall suppose in what follows that, as is the case with ψ, x and y, all the other pertinent variables of the problem have been rendered dimensionless by means of L, U_∞ and some characteristic temperature difference ΔT.) Hence, we shall suppose that $\psi(x,y)$ is a known function and, without loss of generality, that $\psi(x,0) = 0$. Moreover, since we are considering systems with a large characteristic Reynolds number,

$$Re = U_\infty L/\nu$$

(where ν is the kinematic viscosity of the fluid), we can simplify the problem still further by restoring to laminar boundary-layer theory, according to which the stream function is given asymptotically by

$$\psi(x,y) = Re^{\frac{1}{2}}[\overline{\psi}(x,\overline{y}) + O(Re^{-\frac{1}{2}})], \quad \text{for} \quad Re \to \infty, \tag{2.1}$$

in the region where the co-ordinate $\overline{y}$, defined by

$$\overline{y} = Re^{\frac{1}{2}}y, \tag{2.2}$$

is $O(1)$, i.e. lies inside the viscous layer. Furthermore, we shall suppose that the function $\overline{\psi}$ in (2.1) does not depend explicitly on Re and that it can be determined in principle from the standard boundary-layer equations (Meksyn 1961).

As for the temperature field, $\theta(x, y)$ say, we shall suppose as is customary that, inside the viscous layer,
$$\theta = \overline{\theta}(x, \overline{y}) + O(Re^{-\frac{1}{2}}),$$

where $\overline{\theta}(x, \overline{y})$ satisfies the equation
$$\overline{P}\overline{\theta} \overset{\text{def}}{=} \frac{\partial \overline{\psi}}{\partial \overline{y}} \frac{\partial \overline{\theta}}{\partial x} - \frac{\partial \overline{\psi}}{\partial x} \frac{\partial \overline{\theta}}{\partial \overline{y}} - \frac{\gamma_0(x)}{Pr} \frac{\partial^2 \overline{\theta}}{\partial \overline{y}^2} = 0, \tag{2.3}$$

in which, of course, the presence of extraneous heat sources has been excluded. The function $\gamma_0(x)$ appearing here is to be set identically equal to unity for planar flows and equal to the (dimensionless) radius of rotation of the heat-transfer surface in the case of axisymmetric flows (Acrivos & Goddard 1965).

Now, since our objective here is to derive expansions of the temperature field for large or small Pr, which will be accurate to terms of $O(Re^{-\frac{1}{2}})$ in Re (although perhaps not uniformly so for all x), equation (2.3) together with the appropriate boundary conditions will serve as our point of departure. In the problem of interest, the temperature assumes prescribed surface values $\theta_s(x)$ at $\overline{y} = 0$ and is taken to vanish far from the surface. Accordingly, we adopt the boundary conditions:
$$\left.\begin{aligned} &\lim_{\overline{y} \to 0} \overline{\theta}(x, \overline{y}) = \theta_s(x), \quad \text{for} \quad x > 0, \\ &\lim_{\overline{y} \to \infty} \overline{\theta}(x, \overline{y}) = 0, \end{aligned}\right\} \tag{2.4}$$

and
$$\lim_{x \to 0} \overline{\theta}(x, \overline{y}) = 0, \quad \text{for} \quad \text{all} \quad \overline{\psi} > 0, \tag{2.5}$$

where the point $x = 0$ will denote the leading edge or forward stagnation point of the surface in question.

In the case of small Pr the set of equations (2.3) to (2.5) is not sufficient to describe forced convection, nor is it permissible in general to overlook the possibility of a singularity at $x = 0$; this matter though will be taken up again in §4 and in Appendix 1. First of all, we consider the expansions for large Pr, which are somewhat simpler in structure.

3. The asymptotic expansion for heat transfer at large *Pr*

To generate the desired asymptotic expansion we shall resort to the technique discussed in our earlier paper (Acrivos & Goddard 1965). That is, we shall postulate at first a Taylor series expansion for the stream function near the surface; then, after introducing the appropriate 'stretching' transformation of the surface-normal co-ordinate, we shall develop formally an asymptotic expansion for the differential operator governing the convection; and, finally, by employing a standard perturbation analysis, we shall derive a corresponding expansion for the temperature field as well as a sequence of differential equations for successively higher-order terms in this expansion.

It should be strongly emphasized at this point that there is no *a priori* guarantee that any such formal expansion technique will yield asymptotic expansions which will remain uniformly valid under all circumstances. However, as will be discussed below, it is usually possible in any specific example to determine *a posteriori* the necessary conditions for the validity of at least the first-order correction to the asymptotic heat transfer rate, merely by employing the general closed-form expression to be derived here.

Proceeding then in the manner outlined above, we suppose, for generality, that the function $\bar{\psi}(x,\bar{y})$ of (2.1) has an expansion near $\bar{y}=0$ of the form

$$\bar{\psi}(x,\bar{y}) = \bar{\psi}_n(x)\,\bar{y}^n + \bar{\psi}_p(x)\,\bar{y}^p + o(\bar{y}^p), \tag{3.1}$$

where p and n, in general equal to 3 and 2, respectively, are integers such that $p > n \geqslant 0$. Next, with the transformation (cf. Morgan *et al.* 1956)

$$\bar{y}_n = (Pr)^{1/(n+1)}\,\bar{y} = (Pr)^{1/(n+1)}\,(Re)^{\frac{1}{2}}\,y \tag{3.2}$$

we can 'expand' the operator $\bar{P}$ of (2.3) in the form

$$\bar{P} = (Pr)^{(1-n)/(1+n)}[\bar{P}_0 + (Pr)^{-r}\bar{P}_1 + o(Pr^{-r})], \tag{3.3}$$

where $$r = (p-n)/(n+1) \quad (>0),$$

and $\bar{P}_0$ and $\bar{P}_1$ are the differential operators

$$\left.\begin{aligned} \bar{P}_0 &= n\psi_n y^{n-1}\partial/\partial x - \psi_n' y^n \partial/\partial y - \gamma_0\,\partial^2/\partial y^2 \\ \text{and}\qquad \bar{P}_1 &= p\psi_p y^{p-1}\partial/\partial x - \psi_p' y^p \partial/\partial y. \end{aligned}\right\} \tag{3.4}$$

To simplify the notation in equation (3.4), we have written ψ_n and ψ_p for $\bar{\psi}_n$ and $\bar{\psi}_p$, respectively, and y for $\bar{y}_n$; also we have inserted primes to denote differentiation with respect to x.

Now, in view of the form of (3.3), we assume that $\bar{\theta}$ has the asymptotic expansion

$$\bar{\theta}(x,\bar{y};\,Pr) = \bar{\theta}_0(x,\bar{y}_n) + (Pr)^{-r}\bar{\theta}_1(x,\bar{y}_n) + o(Pr^{-r}), \tag{3.5}$$

where $\bar{\theta}_0$ and $\bar{\theta}_1$ do not depend explicitly on Pr. Then, after substituting equations (3.4) and (3.5) into equations (2.3)–(2.4) and equating coefficients of like powers of Pr, we find that the functions $\bar{\theta}_0$ and $\bar{\theta}_1$ must satisfy the differential equations

$$\left.\begin{aligned} &\bar{P}_0\bar{\theta}_0 = 0, \\ &\bar{P}_0\bar{\theta}_1 = -\bar{P}_1\bar{\theta}_0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\text{and the boundary conditions} \\ &\lim_{y\to 0}\bar{\theta}_i(x,y) = \begin{cases}\theta_s(x) & (i=0)\\ 0 & (i=1,\ldots),\end{cases} \\ &\lim_{y\to\infty}\bar{\theta}_i(x,y) = 0 \quad (i=0,1,\ldots) \end{aligned}\right\} \tag{3.6}$$

where, once again, we have replaced $\bar{y}_n$ by y. Similarly, by retaining only the first term of (3.1), we have from equation (2.5) that

$$\lim_{x\to 0}\bar{\theta}_i(x,y) = 0,\, i = 0,1,\ldots, \quad \text{for all} \quad \psi_n(x)\,y^n > 0, \tag{3.7}$$

which completes our formulation of the expansion problem. Without dwelling at this point on questions regarding the validity of the above procedure, we consider now the actual solution for the first 'correction' term in (3.5).

It is evident first of all that the equations for $\bar{\theta}_i$ in (3.6) and (3.7) are special cases of the system:

$$\left.\begin{aligned} &\bar{P}_0\theta = q \\ \text{with}\quad &\lim_{y\to 0}\theta = h \quad (\text{for } x > 0), \\ &\lim_{y\to\infty}\theta = 0, \\ &\lim_{x\to 0}\theta = 0 \quad (\text{for } \psi_n y^n > 0), \end{aligned}\right\} \tag{3.8}$$

where q may be considered a known function of x and y, and h a known function of x. It has already been shown earlier however (Acrivos & Goddard 1965), that the form of the differential operator $\bar{P}_0$ in (3.4) can be greatly simplified by introducing as new independent variables

$$z = [n\psi_n(x)]^{1/n}y, \quad t = \int_0^x [n\psi_n(s)]^{1/n}\gamma_0(s)\,ds \tag{3.9}$$

and that the general solution to (3.8) can be expressed as

$$\theta = \Lambda(t,z) + \chi(t,z), \tag{3.10}$$

where
$$\chi(t,z) = \int_0^t \left[\frac{\partial G_0}{\partial z}(t,z;\, t^*z^*)\right]_{z^*=0} h(t^*)\,dt^* \tag{3.11}$$

and
$$\Lambda(t,z) = \int_{t^*=0}^t \int_{z^*=0}^\infty G_0(t,z;\, t^*,z^*)\,J(t^*)\,q(t^*,z^*)\,dz^*\,dt^*.$$

The functions h and q are those appearing in (3.8), while

$$J\,(t) = \frac{\partial(x,y)}{\partial(t,z)} = \frac{1}{\gamma_0[n\psi_n]^{1/n}} \tag{3.12}$$

is the Jacobian of the transformation (3.9), and $G_0(t,z;\, t^*,z^*)$ is the appropriate Green's function for the differential operator $\bar{P}_0$, all being expressed as functions of z and t. Thus, we can employ here the expression for $G_0(t,z;t^*,z^*)$ given by Sutton (1943) and by Acrivos & Goddard (1965) to write down immediately the solutions to the problem at hand. In particular, we obtain formally for the first two terms of (3.5)

$$\left.\begin{aligned} &\bar{\theta}_0 = \chi(t,z) \\ \text{with}\quad &h(t) \equiv \theta_s(x) \end{aligned}\right\} \tag{3.13}$$

$$\left.\begin{aligned} \text{and}\quad &\bar{\theta}_1 = \Lambda(t,z) \\ \text{with}\quad &q(t,z) \equiv -\bar{P}_1\bar{\theta}_0 \end{aligned}\right\} \tag{3.14}$$

which follow by inspection of (3.6), (3.7), (3.8) and (3.10).

Hence, for a given surface-temperature variation $\theta_s(x)$, we have at our disposal a general formula enabling us to compute in principle the first correction term to the asymptotic heat transfer rate. As was done earlier, however (Acrivos & Goddard 1965), we can, without loss of generality, put our result in a more explicit and useful form by applying it to the simpler problem in which the surface temperature $\theta_s(x)$ varies as a step-function

$$h(t) = \theta_s(x) = H(t-t^*) = H(x-x^*), \tag{3.15}$$

$H(s)$ denoting the Heaviside function. This procedure will yield 'fundamental' solutions to (3.6), $\bar{\theta}_0(x,y;\, x^*)$ and $\bar{\theta}_1(x,y;\, x^*)$, say, which vanish identically for $x < x^*$ and from which we can construct the solutions for more general variations of the surface temperature by invoking the superposition principle for linear equations. As pointed out before (Acrivos & Goddard 1965), the first of these fundamental solutions is simply

$$\bar{\theta}_0(x, y;\, x^*) = \frac{\Gamma(1/(n+1), \zeta^2)}{\Gamma(1/(n+1))} \quad (\text{for } \tau \geqslant 0), \tag{3.16}$$

where

$$\zeta = \frac{z^{\frac{1}{2}(n+1)}}{(n+1)\sqrt{\tau}}, \quad \tau = t - t^*, \quad t^* = t(x^*),$$

$\Gamma(\nu)$ denotes the gamma function, and where $\Gamma(\nu, s)$ is the complement of the incomplete gamma function (with $\Gamma(\nu, 0) = \Gamma(\nu)$).

With $\bar{\theta}_0$ given by (3.16), we can now derive a corresponding explicit expression for the function $\bar{\theta}_1$ of (3.14). First of all, we note that, in terms of the co-ordinates z and t, the operator $\bar{P}_1$ of (3.4) becomes

$$\bar{P}_1 = J^{-1}[pR(t)\, z^{p-1}(\partial/\partial t) - R'(t)\, z^p(\partial/\partial z)] \tag{3.17}$$

where

$$R(t) = \psi_p(x)/[n\psi_n(x)]^{p/n},$$

and where the prime denotes differentiation with respect to t. It follows then that the function q of (3.14) can be expressed as

$$q = q(t, z;\, t^*) = -\frac{(n+1)^{n/(n+1)}\, A_n^{(p)}(\tau, t^*)}{J(t)\, \Gamma(1/(n+1))\, [(n+1)\,\tau]^{1/(n+1)}}\, e^{-\zeta^2}\, [(n+1)\,\zeta]^{2p/(n+1)}$$

with

$$A_n^{(p)}(\tau, t^*) = \partial[\tau^{p/(n+1)}\, R(\tau + t^*)]/\partial\tau. \tag{3.18}$$

Finally, by making use of equations (4.16) and (A 1.4) of the previous article (Acrivos & Goddard 1965) we can show that

$$\bar{\theta}_1 = \bar{\theta}_1(x, y;\, x^*) = S_n^{(p)}(t, z;\, t^*),$$

where $S_n^{(p)}$ is given by

$$S_n^{(p)} = -\left[\frac{(n+1)^{p/(n+1)}}{\Gamma(1/(n+1))}\right]^2 \Gamma\left(\frac{p+2}{n+1}\right) z e^{-\zeta^2} \int_0^1 \lambda^{1/(n+1)}\, (1-\lambda)^{(p-n)/(n+1)},$$

$$A_n^{(p)}(\lambda\tau;\, t^*) \exp\left[-\lambda\zeta^2/(1-\lambda)\right] {}_1F_1\left(\frac{p+2}{n+1};\ \frac{n+2}{n+1};\ \frac{\lambda\zeta^2}{1-\lambda}\right) d\lambda. \tag{3.19}$$

$A_n^{(p)}$ is the function defined by (3.18), and ${}_1F_1$ denotes a confluent hypergeometric function.

Thus, for a given surface temperature distribution, we can use equation (3.19), together with the linear superposition principle to compute the first correction term to the asymptotic temperature profile for $Pr \to \infty$. Of somewhat more interest, however, is the corresponding contribution to the heat-transfer rate, for which an asymptotic series can be derived readily from (2.2), (3.2), (3.5) and (3.9). Expressing this in terms of a 'fundamental' Nusselt number for heat transfer, we have

$$Nu(x;\, x^*) \overset{\text{def}}{=} -\left(\frac{\partial\theta}{\partial y}\right)_{y=0} \equiv -(Re)^{\frac{1}{2}}\, (Pr)^{1/(n+1)} \left(\frac{\partial\bar{\theta}}{\partial\bar{y}_n}\right)_{\bar{y}_n=0}$$

$$= -(Re)^{\frac{1}{2}}\, (Pr)^{1/(n+1)}\, [n\psi_n(x)]^{1/n} \left[\frac{\partial\bar{\theta}_0}{\partial z} + (Pr)^{-r}\frac{\partial\bar{\theta}_1}{\partial z} + o(Pr^{-r})\right]_{z=0} \tag{3.20}$$

to terms of $O(Re^{-\frac{1}{2}})$.

The first term in (3.20), already well known, is given simply by

$$-\left(\frac{\partial\bar{\theta}_0}{\partial z}\right)_{z=0} = \frac{(n+1)^{n/(n+1)}}{\Gamma(1/(n+1))\,[(n+1)\,\tau]^{1/(n+1)}}, \tag{3.21}$$

whereas the second term, which constitutes the final and most important contribution of this section of our paper, becomes

$$-\left(\frac{\partial\bar{\theta}_1}{\partial z}\right)_{z=0} = \left[\frac{(n+1)^{p/(n+1)}}{\Gamma(1/(n+1))}\right]^2 \Gamma\left(\frac{p+2}{n+1}\right)\int_0^1 \lambda^{1/(n+1)}(1-\lambda)^{(p-n)/(n+1)} A_n^{(p)}(\lambda\tau, t^*)\,d\lambda. \tag{3.22}$$

The steps leading from equation (3.19) to (3.22) have been presented earlier (Acrivos & Goddard 1965).

In closing here we should point out that, while it appears difficult to formulate rigorously the most general conditions for the uniform validity of (3.20), it seems logical to suppose that the correctness of our expansion should depend mainly on the analytic character of $\bar{\psi}_n(x)$ and $\bar{\psi}_p(x)$ in (3.1). For example, we would certainly not expect (3.20) to remain valid in the neighbourhood of a point $x = x_0$ where $\bar{\psi}_p/\bar{\psi}_n \to \infty$ in a strongly singular fashion; in fact, we can show rather easily, by (3.17), (3.18), and (3.22), that if

$$\bar{\psi}_n \sim a(x_0-x)^{\nu}, \quad \bar{\psi}_p \sim b(x_0-x)^{\mu}$$

for $x \to x_0$ ($x^* < x < x_0$), where a, b, ν and μ are constants and $\nu > 0$, we must require that

$$(\mu-1)\,n+\mu > (\nu-1)\,p+\nu$$

if the ratio of the second term of (3.20) to the first is to remain bounded near $x = x_0$. This condition reduces to

$$3\mu+1 > 4\nu$$

when, as is usually the case, $n = 2$ and $p = 3$. In contrast, the expansion for the total heat flux over $0 \leqslant x \leqslant x_0$, which is obtained by integrating (3.20), should remain valid to terms of $O(Pr^{-r})$, as long as

$$\mu n(n+1) + (n+\nu) + np(1-\nu) > 0$$

which becomes $6\mu+8 > 5\nu$ when $n = 2$ and $p = 3$. Finally, it should also be stressed that, in certain stagnant-fluid regions, where ψ_n and all higher-order coefficients in (3.1) vanish or near points of surface temperature discontinuity we should not expect (3.20) to provide in general a valid approximation to the heat-transfer rate, since in such cases equation (2.3), i.e. the boundary-layer form of the exact energy equation, ceases to apply. Thus, it should be remembered that, by itself, (3.20) will generally represent a valid approximation to the heat-transfer rate only if the contribution of the terms which were neglected in (2.3) remain much smaller than the term $O(Pr^{-r})$ that was retained in the expansion of (3.5).

4. Expansions for small *Pr*

The mathematical technique outlined in the preceding section can also be applied with some modification to the problem of deriving small *Pr* expansions for heat transfer. As stated in the introduction, this problem has already been considered by Morgan *et al.* (1958). However, our approach will differ somewhat from theirs in that we shall treat the expansion problem explicitly as a *singular-perturbation* scheme involving *two* distinct expansions with different regions of validity, namely, an 'outer expansion' for the temperature field in the region *outside* the momentum layer, and an 'inner-expansion', valid *inside* this layer. (For a discussion of this technique and its application to other problems in fluid mechanics, see Proudman & Pearson 1957 or Taylor & Acrivos 1964.) To some extent, our treatment will facilitate a systematic derivation of additional correction terms to the results given previously.

4.1. *The outer expansion*

In order to derive this expansion, we have to consider first the analytic form of the stream function in the region outside the momentum boundary layer. Now, according to boundary-layer theory we can state formally that, in this region,

$$\psi(x,y) = \Psi^{(0)}(x,y) + (Re)^{-\frac{1}{2}}\Psi^{(1)}(x,y) + O(Re^{-1}), \tag{4.1}$$

where $\Psi^{(0)}$ is the stream function for the exterior potential flow and $\Psi^{(1)}$ is the perturbation on this flow due to the 'displacement' by the viscous layer (Morgan *et al.* 1958). Furthermore, since the limiting form of (4.1), for $y \to 0$, must match with the limiting form of (2.1), for $\bar{y} \to \infty$, to terms of the same order in Re, this requires that

$$\Psi^{(0)}(x,0) = 0$$

and

$$\Psi_0^{(1)}(x) \overset{\text{def}}{=} \Psi^{(1)}(x,0) = \lim_{\bar{y}\to\infty}[\bar{\psi}(x,\bar{y}) - \bar{y}\Psi_1^{(0)}(x)]. \tag{4.2}$$

Here $\Psi_1^{(0)}(x)$ (assumed to be non-negative) is the first coefficient of the Taylor series expansion of $\Psi^{(0)}$, say

$$\Psi^{(0)}(x,y) = y\Psi_1^{(0)}(x) + y^2\Psi_2^{(0)}(x) + \dots \tag{4.3}$$

and $\Psi_0^{(1)}(x)$ is the first coefficient in the expansion of $\Psi^{(1)}$, the latter being generally, of the form,

$$\Psi^{(1)}(x,y) = \Psi_0^{(1)}(x) + y\Psi_1^{(1)}(x) + \dots. \tag{4.4}$$

With the behaviour of the stream function thus specified, and under the assumption that Pe is large, we can now derive a *double* expansion of the temperature field for the region outside the momentum boundary layer by employing a small Pr expansion plus a boundary-layer analysis based on Pe, of the type discussed by Acrivos & Goddard (1965). Thus

$$\theta = \theta_0^{(0)}(x,y_1) + (Pr)^{\frac{1}{2}}\theta_0^{(1)}(x,y_1) + (Pr)\,\theta_0^{(2)}(x,y_1) + O(Pr^{\frac{3}{2}}) + (Pe)^{-\frac{1}{2}}[\theta_1^{(0)}(x,y_1) + O(Pr^{\frac{1}{2}})] + O(Pe^{-1}) \tag{4.5}$$

for $Pe \to \infty$, $Pr \to 0$, where the $\theta_j^{(i)}$ depend only on x and on the variable

$$y_1 \overset{\text{def}}{=} (Pe)^{\frac{1}{2}}y \equiv (Pr)^{\frac{1}{2}}\bar{y} \tag{4.6}$$

(cf. equation (2.2)). This requires that the functions $\theta_0^{(i)}(x,y_1)$ $i = 0,1,2,\dots$ satisfy the differential equation

$$P_0^{(0)}\theta_0^{(i)} = \begin{Bmatrix} 0 & \text{for} \quad i = 0 \\ -P_0^{(1)}\theta_0^{(i-1)} & \text{for} \quad i = 1,2,\dots, \end{Bmatrix} \tag{4.7}$$

where $P_0^{(0)}$ and $P_0^{(1)}$ are the differential operators:

$$\left.\begin{aligned} P_0^{(0)} &= \Psi_1^{(0)}(x)\,\partial/\partial x - \Psi_1^{(0)\prime}(x)\,y_1\,\partial/\partial y_1 - \gamma_0\,\partial^2/\partial y_1^2, \\ P_0^{(1)} &= -\Psi_0^{(1)\prime}(x)\,\partial/\partial y_1. \end{aligned}\right\} \tag{4.8}$$

Again the functions $\Psi_1^{(0)}$ and $\Psi_0^{(1)}$ are the leading coefficients in the expansions (4.3) and (4.4), and the primes denote differentiation with respect to x.

On inspection of (4.5) it becomes evident that the function $\theta_0^{(0)}$ represents the limiting form of the temperature field for $Pr \to 0$ and $Pe \to \infty$ and that the $\theta_0^{(i)}$ ($i = 1, 2, \ldots$), are higher-order correction terms in Pr which are due to the 'out-flow' produced by the viscous layer. In contrast, the term $\theta_1^{(0)}$ in (4.5) represents a correction for the effect of surface curvature, the derivation of which will be deferred to Appendix 1. It should be noted that the terms indicated explicitly in the expansion (4.5) are subject to *a posteriori* verification since they will be derived below, whereas the use of O-symbols for terms not appearing explicitly is at this point only a formalism, which we have employed here temporarily to elucidate the structure of the perturbation scheme. Without a more detailed analysis of these higher-order terms it will be necessary to replace the O-symbols by the appropriate o-symbols in the final result for the heat transfer rate to be derived later on.

Now, for the boundary-value problem under consideration, we require that the functions $\theta_j^{(i)}(x, y_1)$ satisfy the conditions

$$\left.\begin{aligned} &\lim_{y_1 \to \infty} \theta_j^{(i)}(x, y_1) = 0, \\ &\lim_{x \to 0} \theta_j^{(i)}(x, y_1) = 0 \quad \text{for all} \quad \Psi_1^{(0)}(x)\,y_1 > 0 \end{aligned}\right\} \tag{4.9}$$

for $i, j = 0, 1, 2, \ldots$. However, since the expansion (4.5) is valid only in the region outside the viscous layer ($\bar{y} \gg 1$), the appropriate boundary conditions on the $\theta_j^{(i)}$ at $y_1 = 0$ must be determined by matching (4.5) with an inner expansion whose form is to be considered next.

4.2. *The inner expansion*

To derive the inner expansion we shall adopt here a method similar to that first employed by Morgan *et al.* (1958). Before proceeding, however, we wish to emphasize at the outset that the expansion thus obtained will be valid only for smooth differentiable surface temperature variations. This is so because, aside from the usual defect of the boundary-layer approximations due to the omission of 'diffusion' terms involving $\partial^2/\partial x^2$, there is a second, more serious defect inherent in the small Pr-expansion which results from neglecting, as $Pr \to 0$, 'convection' terms involving $\partial/\partial x$. As it turns out, this is tantamount to assuming that, for $Pr \to 0$, the temperature is essentially constant across the viscous layer, an assumption which appears plausible since this layer is asymptotically much thinner than the thermal layer. Nevertheless, although this condition is realized over most of the heat transfer surface, the reverse will always be true, no matter how small Pr, at the leading edge of a thermal layer (i.e. at a *discontinuity* in surface temperature) if this is located downstream from the leading edge of the viscous layer. In fact, the 'large Pr' expansions discussed in (4.2) should become applicable near such points, as is indicated in § 5.1 below.

With the above reservation in mind we find, by a slight extension of the technique of Morgan *et al.* to account for surface curvature, that inside the viscous layer

$$\theta = \bar{\theta}_0^{(0)}(x,\bar{y}) + (Pr)^{\frac{1}{2}}\bar{\theta}_0^{(1)}(x,\bar{y}) + (Pr)\,\bar{\theta}_0^{(2)}(x,\bar{y}) + (Pr)^{\frac{3}{2}}\bar{\theta}_0^{(3)}(x,\bar{y}) + O(Pr^2) + (Pe)^{-\frac{1}{2}}[(Pr)^{\frac{1}{2}}\bar{\theta}_1^{(0)}(x,\bar{y}) + O(Pr)] + O(Pe^{-1}), \quad (4.10)$$

where the functional coefficients $\bar{\theta}_j^{(i)}(x,\bar{y})$, which in contra-distinction to (4.5) have been provided with overbars, are functions of x and $\bar{y}$ only. The terms here of order zero in Pe represent merely an expansion in Pr for the function $\bar{\theta}$ of (2.3), the functional coefficients of which, $\bar{\theta}_0^{(i)}$, must satisfy the following equations, derived essentially from (2.3)

$$\gamma_0 \frac{\partial^2 \bar{\theta}_0^{(i)}}{\partial \bar{y}^2} = \begin{cases} 0, & \text{for } i = 0, 1, \\ \dfrac{\partial \bar{\psi}}{\partial \bar{y}}\dfrac{\partial \bar{\theta}_0^{(i-2)}}{\partial x} - \dfrac{\partial \bar{\psi}}{\partial x}\dfrac{\partial \bar{\theta}_0^{(i-2)}}{\partial \bar{y}} & \text{for } i = 2, 3, \ldots, \end{cases} \quad (4.11)$$

where $\bar{y}$ is the variable of (2.2). The appropriate differential equation for the function $\bar{\theta}_1^{(0)}$, the correction for surface curvature, is given in Appendix 1.

Again, as with (4.5), we should emphasize that the use of O-symbols for terms not appearing explicitly in (4.10) is merely a formal device, and that these terms will be replaced by the appropriate o-symbols later on in the analysis.

Finally, the boundary conditions on the $\bar{\theta}_j^{(i)}$ at $\bar{y} = 0$ follow directly from (4.10) and (2.4),

$$\lim_{\bar{y}\to 0} \bar{\theta}_j^{(i)} = \begin{cases} \theta_s(x) & \text{for } i = j = 0, \\ 0 & \text{otherwise}, \end{cases} \quad (4.12)$$

while the necessary conditions at $\bar{y} = \infty$ can be determined, as we now show, by matching (4.10) to the outer expansion (4.5).

4.3. *Matching of the expansions*

To derive the necessary additional boundary conditions for the functions in the expansions (4.5) and (4.10), we shall require that the limiting form of (4.10) for $\bar{y} \to \infty$ join analytically with the limiting form of (4.5) for $y_1 \to 0$ *to terms of the same order in both Pr and Pe.* For this purpose, it will be necessary to attribute an analytic character to the functions $\theta_j^{(i)}(x, y_1)$, in which case they may be assumed to have Taylor series of the form

$$\theta_j^{(i)}(x,y_1) = \theta_j^{(i)}(x,0) + y_1 \frac{\partial \theta_j^{(i)}}{\partial y_1}(x,0) + \frac{y_1^2}{2}\frac{\partial^2 \theta_j^{(i)}}{\partial y_1^2}(x,0) + \ldots, \quad (4.13)$$

with finite differential coefficients. With this assumption, the limiting form of the outer expansion (4.5) near $y_1 = 0$ is readily found to be

$$\theta = \Big\{\theta_0^{(0)}(x,0) + Pr^{\frac{1}{2}}\left[\bar{y}\frac{\partial \theta_0^{(0)}}{\partial y_1}(x,0) + \theta_0^{(1)}(x,0)\right] + Pr\left[\frac{\bar{y}^2}{2}\frac{\partial^2\theta_0^{(0)}}{\partial y_1^2}(x,0) + \bar{y}\frac{\partial \theta_0^{(1)}}{\partial y_1}(x,0) + \theta_0^{(2)}(x,0)\right] + O(Pr^{\frac{3}{2}})\Big\} + (Pe)^{-\frac{1}{2}}\{\theta_1^{(0)}(x,0) + O(Pr^{\frac{1}{2}})\} + O(Pe^{-1})$$

on making use of (4.6). Hence, by matching this series in the prescribed manner to that of (4.10) we obtain that

$$\left.\begin{aligned}
\theta_0^{(0)}(x,0) &= \lim_{\bar{y}\to\infty} \bar{\theta}_0^{(0)}(x,\bar{y}),\\
\theta_0^{(1)}(x,0) &= \lim_{\bar{y}\to\infty}\left[\bar{\theta}_0^{(1)}(x,\bar{y}) - \bar{y}\,\frac{\partial\theta_0^{(0)}}{\partial y_1}(x,0)\right],\\
\theta_0^{(2)}(x,0) &= \lim_{\bar{y}\to\infty}\left[\bar{\theta}_0^{(2)}(x,\bar{y}) - \bar{y}\,\frac{\partial\theta_0^{(1)}}{\partial y_1}(x,0) - \frac{\bar{y}^2}{2}\,\frac{\partial^2\theta_0^{(0)}}{\partial y_1^2}(x,0)\right],\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
\text{and}\qquad \theta_1^{(0)}(x,0) &= 0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\}\tag{4.14}$$

The further requirement that the limits on the right-hand side of these equations exist provides all the necessary conditions on the $\bar{\theta}_j^{(i)}$ at $\bar{y} = \infty$ and simultaneously yields boundary values for the $\theta_j^{(i)}$ at $y_1 = 0$.

4·4. *The solution for the inner region*

The solution for the terms indicated explicitly in (4.10) follows now in a rather straightforward manner due to the simplicity of the differential equations of (4.11). Thus, the first equations of (4.11) and (4.12) imply that

$$\bar{\theta}_0^{(0)}(x,\bar{y}) \equiv \theta_s(x) \tag{4.15}$$

up to an additive, linear, homogeneous function of $\bar{y}$ which, however, must be taken as zero in order for the limit on the right-hand side of the first equation of (4.14) to exist. Whence, the boundary condition on $\theta_0^{(0)}(x,y_1)$ at $y_1 = 0$ follows immediately,

$$\theta_0^{(0)}(x,0) = \theta_s(x). \tag{4.16}$$

This result together with equations (4.7) and (4.9) suffices for the determination of $\theta_0^{(0)}(x,y_1)$, which, on account of the second equation of (4.14), can be used in turn to yield the necessary conditions on $\bar{\theta}_0^{(1)}(x,\bar{y})$ as $\bar{y}\to\infty$ as well as the boundary value $\theta_0^{(1)}(x,0)$. This procedure can be repeated of course for the higher-order terms.

Without elaborating on the details, which are essentially algebraic in nature, we shall merely summarize the results which one obtains by continuing in the above fashion. Thus, by considering the remaining terms in (4.5) and (4.10) of order zero in Pe, we find for the terms of $O(Pr^{\frac{1}{2}})$ that

$$\bar{\theta}_0^{(1)}(x,\bar{y}) = \bar{y}\,\frac{\partial\theta_0^{(0)}}{\partial y_1}(x,0) \tag{4.17}$$

with

$$\theta_0^{(1)}(x,0) = 0,$$

and for those of $O(Pr)$,

$$\bar{\theta}_0^{(2)}(x,\bar{y}) = \bar{y}\,\frac{\partial\theta_0^{(1)}}{\partial y_1}(x,0) + \frac{\bar{y}^2}{2}\,\frac{\partial^2\theta_0^{(0)}}{\partial y_1^2}(x,0) + \frac{\theta_s'(x)}{\gamma_0(x)}\int_0^{\bar{y}} \Delta\bar{\psi}(x,s)\,ds \tag{4.18}$$

with

$$\theta_0^{(2)}(x,0) = \frac{\theta_s'(x)}{\gamma_0(x)}\int_0^{\infty} \Delta\bar{\psi}(x,\bar{y})\,ds, \tag{4.18a}$$

where we have defined

$$\Delta\bar{\psi}(x,\bar{y}) = \bar{\psi}(x,\bar{y}) - \bar{y}\Psi_1^{(0)}(x) - \Psi_0^{(1)}(x). \tag{4.19}$$

Finally, for the terms of $O(Pr^{\frac{3}{2}})$, we obtain

$$\bar{\theta}_0^{(3)}(x,\bar{y}) = \sum_{k=1}^{3} \frac{\bar{y}^k}{k!}\frac{\partial^k\theta_0^{(3-k)}}{\partial y_1^k}(x,0) + \frac{1}{\gamma_0(x)}\left\{\frac{\partial^2\theta_0^{(0)}}{\partial x\,\partial y_1}(x,0)\int_0^{\bar{y}} s\Delta\bar{\psi}(x,s)\,ds + \frac{\partial}{\partial x}\left[\frac{\partial\theta_0^{(0)}}{\partial y_1}(x,0)\int_0^{\bar{y}}\int_s^\infty \Delta\bar{\psi}(x,s_1)\,ds_1 ds\right]\right\}. \quad (4.20)$$

We could have written down also the boundary value $\theta_0^{(3)}(x,0)$, but this would have been superfluous for the present purposes. Instead, of more interest are the $\bar{y}$-derivatives of $\bar{\theta}_0^{(i)}$ at $\bar{y} = 0$. Those of $\bar{\theta}_0^{(1)}$ and $\bar{\theta}_0^{(2)}$ can be obtained immediately from (4.17) and (4.18), while, because of (4.20),

$$\left(\frac{\partial\bar{\theta}_0^{(3)}}{\partial\bar{y}}\right)_{\bar{y}=0} = \frac{\partial\theta_0^{(2)}}{\partial y_1}(x,0) + \frac{1}{\gamma_0(x)}\frac{\partial}{\partial x}\left[\frac{\partial\theta_0^{(0)}}{\partial y_1}(x,0)\int_0^\infty \Delta\bar{\psi}(x,\bar{y})\,d\bar{y}\right]. \quad (4.21)$$

In addition it is shown in Appendix 1 that the term of $O(Pe^{-\frac{1}{2}})$ in (4.5) is related to the term of $O(Pr^{\frac{1}{2}}Pe^{-\frac{1}{2}})$ in (4.10) by

$$\bar{\theta}_1^{(0)}(x,\bar{y}) = \bar{y}\frac{\partial\theta_1^{(0)}}{\partial y_1}(x,0), \quad (4.22)$$

while, on account of (4.14), the boundary value $\theta_1^{(0)}(x,0)$ must be zero. In closing here, we should note further that (4.18) is equivalent to the result already given by Morgan *et al.* (1958) for planar flows.

4.5. *The solution for the outer region*

One can now determine the coefficients $\theta_j^{(i)}$ in the outer expansion (4.5) using as boundary conditions (4.17), (4.18) and the last equation of (4.14). In fact, we have only to observe that the equations of (4.7), and the operator $P_0^{(0)}$ of (4.8) are, respectively, special cases ($n = 1$) of the system (3.8), and of the operator $\bar{P}_0$ of (3.4). Therefore, exactly as was done for the coefficients in the expansions of §3, we can again employ the Green's function of Acrivos & Goddard (1965) (with $n = 1$, now) in conjunction with equations (3·11) to yield formal solutions for the $\theta_j^{(i)}$.

Adopting then the same technique as in §3, we shall suppose next that the surface temperature is given by the step-function of (3.15) and shall thereby derive 'fundamental' solutions to the boundary-value problem under consideration. Thus, the leading term in (4.13) becomes merely the special case of (3.16) corresponding to $n = 1$,

$$\left.\begin{aligned} \theta_0^{(0)}(x,y_1) &= \Gamma(\tfrac{1}{2},\zeta^2)/\Gamma(\tfrac{1}{2}) \equiv \operatorname{erfc}\zeta \\ \text{with}\quad (\partial\theta^{(0)}/\partial z)_{z=0} &= -(\pi\tau)^{-\frac{1}{2}}, \end{aligned}\right\} \quad (4.23)$$

where erfc denotes the complementary error function. Also, the variables

$$\zeta = \tfrac{1}{2}z/\tau^{\frac{1}{2}} \quad (4.24)$$

and τ are those given by (3.16), and the variables z and t are to be obtained from equations (3.9) by taking $n = 1$ and $\psi_1 = \Psi_1^{(0)}$. In other words,

$$z = \Psi_1^{(0)}(x)\,y_1, \quad t = \int_0^x \Psi_1^{(0)}(s)\,\gamma_0(s)\,ds, \quad (4.25)$$

where $\Psi_1^{(0)}$ is, of course, the functional coefficient appearing in (4.3) and (4.8).

Having thus found $\theta_0^{(0)}$ it is possible to specify the corresponding non-homogeneous term in the differential equation of (4.7) for $\theta_0^{(1)}$, and to proceed with the solution for this function. In particular, it follows from (4.7), (4.9) and (4.17) that $\theta_0^{(1)}$ is given by the integral Λ of (3.11) with

$$q = -P_0^{(1)}\theta_0^{(0)}$$

in which the operator $P_0^{(1)}$, defined by (4.8), can be expressed simply as

$$P_0^{(1)} = -\frac{1}{J(t)}\frac{dR(t)}{dt}\frac{\partial}{\partial z}, \tag{4.26}$$

where z and t are given by (4.25), J is the Jacobian of (3.12), and

$$R(t) = \Psi_0^{(1)}(x), \tag{4.27}$$

$\Psi_0^{(1)}$ being the first term in the expansion (4.4). It becomes evident therefore that (4.26) and (4.27) are merely special cases of (3.17) with $p = 0$. In addition, since the expression for $\theta_0^{(0)}$ as given by (4.23) is a special case of (3.16), it follows by taking

$$A_1^{(0)}(\tau, t^*) \equiv \partial R(\tau + t^*)/\partial\tau \tag{4.28}$$

which is in turn a special case of (3.18), that

$$\theta_0^{(1)} = S_1^{(0)}(t, z;\, t^*),$$

where $S_1^{(0)}$ is the integral (3.19), with $n = 1, p = 0$. Moreover, since the confluent hypergeometric function which occurs in that integral reduces to the form

$${}_1F_1(1;\, \tfrac{3}{2};\, \rho) \equiv \tfrac{1}{2}(\pi/\rho)^{\frac{1}{2}}\, e^{\rho}\, \mathrm{erf}\,(\rho^{\frac{1}{2}})$$

in which erf denotes the error function,

$$\theta_0^{(1)} = S_1^{(0)}(t, z;\, t^*) = -\sqrt{\frac{\tau}{\pi}}\, e^{-\zeta^2}\int_0^1 A_1^{(0)}(\lambda\tau, t^*)\, \mathrm{erf}\left(\zeta\left(\frac{\lambda}{1-\lambda}\right)^{\frac{1}{2}}\right) d\lambda, \tag{4.29}$$

with
$$\left(\frac{\partial\theta_0^{(1)}}{\partial z}\right)_{z=0} = -\frac{1}{\pi}\int_0^1 \left(\frac{\lambda}{1-\lambda}\right)^{\frac{1}{2}} A_1^{(0)}(\lambda\tau, t^*)\, d\lambda, \tag{4.30}$$

where $A_1^{(0)}$ is given by (4.28).

With $\theta_0^{(1)}$ specified now by (4.29), we are in a position to solve the appropriate differential equation (4.7) for the next higher-order term $\theta_0^{(2)}$. Since this function is to satisfy the boundary conditions (4.9) and (4.18), the formal solution can again be written down immediately as a sum of the general integrals in (3.11), i.e.

$$\theta_0^{(2)} = \chi + \Lambda \quad \text{with} \quad q \equiv -P_0^{(1)}\theta_0^{(1)}, \tag{4.31}$$

and $h(t)$ given by (4.18a). However, it will be noted that (4.18a) contains the first derivative of the surface temperature, and, since we have chosen to treat the problem with a surface temperature having the step-function behaviour of (3.15), this derivative will not exist in the ordinary sense, a difficulty already foreseen above. While we can temporarily overcome the mathematical aspect of this difficulty by employing 'generalized' functions, the results so derived will be meaningful ultimately only if interpreted as linear operators, to be applied according to the linear superposition principle in those cases where the surface temperature is differentiable. This serves to re-emphasize the fact that, unless the surface temperature does indeed vary smoothly, the expansions derived here are highly singular in nature. With this understanding, we can express (4.18 a) as

$$\theta_0^{(2)}(x,0) = h(t) = \delta(t-t^*)\,M(t^*), \tag{4.32}$$

where δ denotes the Dirac delta function and

$$M(t) = \Psi_1^{(0)}(x)\int_0^\infty \Delta\psi(x,\bar{y})\,d\bar{y}, \tag{4.33}$$

t and x being related by (4.25). Thus, the first contribution to $\theta_0^{(2)}$ in (4.31) is found by substituting the function $h(t)$ of equation (4.32) into the integral χ of (3.11), yielding (symbolically)

$$\chi = \left[\frac{\partial G_0}{\partial z}(t,z;\,t^*\,z^*)\right]_{z^*=0} M(t^*) = \frac{z}{2(\pi\tau^3)^{\frac{1}{2}}}\,e^{-z^2/4\tau}\,M(t^*)\,H(t-t^*), \tag{4.34}$$

where we have used the appropriate expression for the Green's function G_0, which was given earlier by Acrivos & Goddard (1965).

It can be verified readily that, (4.34) does indeed have the delta-function behaviour at $z=0$ required by (4.32), and that furthermore,

$$\left(\frac{\partial\chi}{\partial z}\right)_{z=0} = M(t^*)\frac{\partial}{\partial t^*}\left(\frac{H(t-t^*)}{[\pi(t-t^*)]^{\frac{1}{2}}}\right). \tag{4.35}$$

As one might have anticipated, the second contribution to $\theta_0^{(2)}$, which is given by the integral Λ of (3.11) with q taken as indicated in (4.31), turns out to have a rather complicated form. However, we are mainly interested in the contribution of this term to the surface-normal derivative of $\theta_0^{(2)}$, for which we can obtain a much simpler result. In particular, if we assume that it is permissible to exchange the order of integration and differentiation with respect to z in (3.11), we find, again using the expression for the Green's function given by Acrivos & Goddard (1965), that

$$\left(\frac{\partial\Lambda}{\partial z}\right)_{z=0} = -\frac{1}{\pi}\sqrt{\frac{\tau}{\pi}}\int_0^1\int_0^1 A_1^{(0)}(\lambda'\tau,t^*)\,A_1^{(0)}(\lambda'\lambda\tau,t^*)\,f(\lambda,\lambda')\,d\lambda'\,d\lambda, \tag{4.36}$$

where $$f(\lambda,\lambda') = \left(\frac{\lambda}{(1-\lambda)(1-\lambda')}\right)^{\frac{1}{2}}\frac{1}{c^2} - \lambda'\left[\frac{d}{ds}(s\tan^{-1}s)\right]_{s=a/b}$$

and $$a = \left(\frac{\lambda(1-\lambda')}{\lambda'(1-\lambda)}\right)^{\frac{1}{2}},\quad b = \frac{1}{\sqrt{\lambda'}},\quad c^2 = a^2+b^2 = \frac{1-\lambda\lambda'}{\lambda'(1-\lambda)}.$$

Thus, in conclusion, we have by (4.31) that

$$(\partial\theta_0^{(2)}/\partial z)_{z=0} = (\partial\chi/\partial z)_{z=0} + (\partial\Lambda/\partial z)_{z=0}, \tag{4.37}$$

where the functions (of t and t^*) on the right-hand side are now completely specified by (4.35) and (4.36).

4.6. *The expansion for the rate of heat transfer*

By means of the results of §§4.1–4.5, we shall now derive the desired asymptotic expansion for the heat-transfer rate. Thus, expressing this as before, in terms of a 'fundamental' Nusselt number, we obtain from the *inner expansion* (4.19) that

$$\begin{aligned} Nu(x;\, x^*) &= -(\partial\theta/\partial y)_{y=0} \\ &= -(Pe)^{\frac{1}{2}}\{[\partial\bar{\theta}_0^{(1)}/\partial\bar{y} + Pr^{\frac{1}{2}}\,\partial\bar{\theta}_0^{(2)}/\partial\bar{y} + Pr\,\partial\bar{\theta}_0^{(3)}/\partial\bar{y} + o(Pr)]_{\bar{y}=0} \\ &\quad + Pe^{-\frac{1}{2}}[\partial\bar{\theta}_1^{(0)}/\partial\bar{y}]_{\bar{y}=0} + o(Pe^{-\frac{1}{2}})\}, \end{aligned}$$

where, in accordance with our earlier remarks regarding the significance of the symbols O we have replaced these by the appropriate o-symbols. Now, by employing (4.17) to (4.21), (4.27) and (4.33) we can express the preceding result as

$$\begin{aligned} F(x;\, x^*) &\overset{\text{def}}{=} (2t/Re)^{\frac{1}{2}} Nu(x;\, x^*)/\Psi_1'^{(0)}(x) = (Pr)^{\frac{1}{2}}\{K_0^{(0)}(t;\, t^*) + Pr^{\frac{1}{2}} K_0^{(1)}(t;\, t^*) \\ &\quad + PrK_0^{(2)}(t;\, t^*) + o(Pr) + Pe^{-\frac{1}{2}} K_1^{(0)}(t;\, t^*) + o(Pe^{-\frac{1}{2}})\} \end{aligned} \tag{4.38}$$

for $Pr \to 0$, $Pe \to \infty$, where

$$\left.\begin{aligned} K_0^{(0)}(t;\, t^*) &= -(2t)^{\frac{1}{2}}(\partial\theta_0^{(0)}/\partial z)_{z=0} \equiv [2/\pi(1-t/t^*)]^{\frac{1}{2}}, \\ K_0^{(1)}(t;\, t^*) &= (2t)^{\frac{1}{2}}[-(\partial\theta_0^{(1)}/\partial z)_{z=0} + R(t)\delta(t-t^*)], \\ K_0^{(2)}(t;\, t^*) &= (2t)^{\frac{1}{2}}\left[-\left(\frac{\partial\theta_0^{(2)}}{\partial z}\right)_{z=0} + \frac{\partial}{\partial t}\left(\frac{M(t)\,H(t-t^*)}{\{\pi(t-t^*)\}^{\frac{1}{2}}}\right)\right], \\ \text{and}\quad K_1^{(0)}(t;\, t^*) &= -(2t)^{\frac{1}{2}}(\partial\theta_1^{(0)}/\partial z)_{z=0}. \end{aligned}\right\} \tag{4.39}$$

With the exception of the last equation, the definitions of the functions appearing on the right-hand side of the equations in (4.39) can be found by referring to (4.30), (4.33) and (4.37). In Appendix 1 an expression is derived giving $K_1^{(0)}$ and, thus, the correction term for surface curvature of $O(Pr^{\frac{1}{2}}Pe^{-\frac{1}{2}})$ in (4.38), which, as is shown there, vanishes identically for planar flows. The factor $(2t)^{\frac{1}{2}}$ has been included arbitrarily in (4.38) and (4.39), in order that, as is to be shown in §5.2 below, the functions $K(t;\, t^*)$ be expressible simply as functions of (t/t^*) in the case of 'similarity' flows.

We can readily deduce now that the term of $O(Pr)$ in (4.38) is in essence equivalent to the result already given by Morgan *et al.* (1958) for planar flows. The additional terms derived here would permit one to compute in principle the heat transfer rate, for $Pr \ll 1$ and $Pe \gg 1$, up to terms which are formally either $O(Pr^{\frac{3}{2}})$, $O(Pe^{-1})$ or $O(Re^{-\frac{1}{2}}) \equiv O(Pr^{\frac{1}{2}}Pe^{-\frac{1}{2}})$ relative to the leading term. Of course, in order to calculate the term $O(Pr^{\frac{1}{2}}Pe^{-\frac{1}{2}})$ we would require the $O(Re^{-\frac{1}{2}})$ term in the expansion of the stream function (2.1) which is not provided generally by the usual laminar 'boundary-layer' approximation.

In closing here, we should summarize the limitations of the small Pr expansion, equation (4.38), for the rate of heat transfer. First of all, just as with the large Pr expansion of the preceding section, both (4.5) and (4.10) are subject to the usual defect of boundary-layer expansions arising from the omission, in arriving at (2.3), of 'diffusion' terms involving $\partial^2/\partial x^2$. As a consequence, the expression for the $O(Pe^{-\frac{1}{2}}Pr^{\frac{1}{2}})$ term ceases to apply in the neighbourhood of a point $x = x^*$, where the surface temperature is discontinuous. Near such a point, the expansion (4.38) would have to be replaced by another expansion, valid in the singular region

$$|x-x^*| = O(Pe^{-\frac{1}{2}}).$$

Thus, whereas the term of $O(Pe^{-\frac{1}{2}}Pr^{\frac{1}{2}})$ in (4.38) vanishes identically for planar flows with smoothly varying surface temperatures, in the case of a discontinuous surface temperature distribution there would be in general a contribution to the total heat transfer rate of $O(Pe^{-\frac{1}{2}})$ relative to the first term. However, in addition to this difficulty, there is also, as indicated earlier, the more serious defect in the small Pr expansion arising from the omission inside the viscous layer of convection terms involving $\partial/\partial x$. Since these terms would become important near a point of discontinuity in the surface temperature, the terms in (4.38) of order zero in Pe would not remain valid in general near such points. Thus, it should be emphasized again that the surface temperature discontinuity at $x = x^*$, which was introduced in this section, is merely a mathematical artifice, it being understood that (4.38) is to be used only in conjunction with differentiable surface temperature distributions.

As a final remark, we note that the appropriate asymptotic expansions for heat transfer with arbitrary surface temperatures $\theta_s(x)$ are to be constructed from the asymptotic expansions of both §§ 3 and 4 according to the superposition formula

$$-\left(\frac{\partial\theta}{\partial y}\right)_{y=0} = \int_0^x Nu(x;x^*)\,d\theta_s(x^*) = \theta_s(0)\,Nu(x;0)+\int_0^x Nu(x;\,x^*)\,\frac{d\theta_s(x^*)}{dx^*}\,dx^*. \tag{4.40}$$

The first integral is of the Stieltjes type, with the second equality holding whenever $\theta_s(x)$ is differentiable. As indicated above, the results of § 4 are valid only when this latter condition is realized.

5. Application to similarity flows

For both large and small Pr, the results derived above can be simplified greatly for 'similar' boundary-layer flows. In this case, the potential-flow velocity at the surface, say $U(x)$, is given by

$$U(x) = t_1^{\beta/2}, \tag{5.1}$$

where

$$t_1 = \int_0^x \Psi_1^{(0)}(s)\,\gamma_0(s)\,ds = \int_0^x U(s)\,\gamma_0^2(s)\,ds \tag{5.2}$$

and β is a constant. The variable t_1, which is identical to that defined by (4.25), has also been used in Görtler's transformation (Meksyn 1961). For flows of the above type, the boundary-layer stream function of (5.1) can then be expressed as

$$\left.\begin{aligned}\bar{\psi}(x,\bar{y}) &= (2t_1)^{\frac{1}{2}}f(\eta),\\ \text{where}\quad \eta = \bar{z}_1/(2t_1)^{\frac{1}{2}},\quad \bar{z}_1 &= \Psi_1^{(0)}(x)\,\bar{y} = \gamma_0(x)\,U(x)\,\bar{y},\end{aligned}\right\} \tag{5.3}$$

and where $f(\eta)$ satisfies the well-known Falkner–Skan equation

$$\left.\begin{aligned}&f''' + ff'' + \beta(1-f'^2) = 0,\\ \text{with}\quad &f' = f = 0 \quad \text{at} \quad \eta = 0,\\ &f' \to 1 \quad \text{as} \quad \eta \to \infty.\end{aligned}\right\} \tag{5.4}$$

One will remember that in the case of planar flow ($\gamma_0 \equiv 1$) the equations of (5.1) to (5.4) govern the symmetric boundary-layer flow past a wedge-shaped surface with included angle $\beta\pi$.

5.1. *Expansions for large Pr*

For a non-separating boundary-layer flow past a solid surface and with a non-zero streamwise pressure gradient, $n = 2$ and $p = 3$ in the stream function expansion of (3.1). One obtains

$$\left.\begin{aligned} \overline{\psi}_2(x) &\equiv \frac{1}{2!}\left(\frac{\partial^2\overline{\psi}}{\partial\bar{y}^2}\right)_{\bar{y}=0} = \frac{1}{2}\frac{[\Psi_1^{(0)}(x)]^2}{(2t_1)^{\frac{1}{2}}}\, f_0'', \\ \overline{\psi}_3(x) &\equiv \frac{1}{3!}\left(\frac{\partial^3\overline{\psi}}{\partial\bar{y}^3}\right)_{\bar{y}=0} = -\frac{\beta}{6}\frac{[\Psi_1^{(0)}(x)]^3}{2t_1}. \end{aligned}\right\} \tag{5.5}$$

The function f is, of course, that of (5.3), and the superscript zero on f will denote henceforth the value of the derivative indicated by an appropriate number of primes at $\eta = 0$.

Equation (3.21) yields immediately the first term in the asymptotic series (3.20), and by the first equation of (5.5), one sees that the pertinent variable t, defined by (3.10) with $n = 2$ and $\psi_2 = \overline{\psi}_2(x)$, is related to t_1 of (5.2) by

$$t = \int_0^x [2\overline{\psi}_2(s)]^{\frac{1}{2}}\,\gamma_0(s)\,ds = \tfrac{2}{3}(f_0'')^{\frac{1}{2}}\,(2t_1)^{\frac{3}{4}}. \tag{5.6}$$

Hence, the second term of (3.20), to be determined by (3.22), can be simplified as follows. Noting first that the function $R(t)$ of (3.17) reduces, by (5.5) and (5.6), to

$$R(t) = -\tfrac{1}{6}\beta\,[2/3(f_0'')^4\,t]^{\frac{1}{3}}$$

we obtain for the function $A_2^{(3)}$ of (3.18) simply

$$A_2^{(3)}(\tau, t^*) = -\tfrac{1}{6}\beta[2/3(f_0'')^4]^{\frac{1}{3}}\,[(\tau+t^*)^{-\frac{1}{3}} - \tfrac{1}{3}\tau(\tau+t^*)^{-\frac{4}{3}}].$$

The substitution of this function into (3.22) gives rise to integrals which can be expressed in terms of the hypergeometric function, according to the relation

$$\begin{aligned} &\int_0^1 (1-\lambda)^{\mu}\,\lambda^{\nu}(\lambda\tau+t^*)^{\gamma}\,d\lambda \\ &\qquad = (t^*)^{\gamma}\,\frac{\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\mu+\nu+2)}\;{}_2F_1(\nu+1, -\gamma;\, \nu+\mu+2;\, -\tau/t^*) \end{aligned} \tag{5.7}$$

valid for $\nu+\mu+2 > -\gamma > 0$, $\tau/t^* > -1$ (Erdelyi *et al.* 1954 H.T.F., vol. 2). Thus, we find for the desired correction term

$$\begin{aligned} (\partial\bar{\theta}_1/\partial z)_{z=0} = &-\tfrac{1}{10}\beta(f_0'')^{-\frac{2}{3}}\,(2t_1)^{-\frac{1}{4}}\,(1+\xi_2^*)^{\frac{1}{3}} \\ &\times\left\{{}_2F_1(\tfrac{4}{3}, \tfrac{1}{3};\, \tfrac{8}{3};\, -\xi_2^*) - \frac{\xi_2^*}{6}\,{}_2F_1(\tfrac{7}{3}, \tfrac{4}{3}; \tfrac{11}{3};\, -\xi_2^*)\right\}, \end{aligned} \tag{5.8}$$

where

$$\xi_2^* \overset{\text{def}}{=} \tau/t^* = t/t^* - 1 = (t_1/t_1^*)^{\frac{3}{4}} - 1 \quad (\geqslant 0),$$

with t_1^* and t^* corresponding to the point $x = x^*$ of the step discontinuity in the surface temperature.

For $\xi_2^* < 1$ (i.e. for $t < 2t^*$) the hypergeometric functions in (5.8) could be calculated in principle by means of the Gaussian series

$$ {}_2F_1(a,b;\,c;\,z) = 1+\frac{a,b}{c}\frac{z}{1!}+\frac{a(a+1)\,b(b+1)}{c\,(c+1)}\frac{z^2}{2!}+\dots, $$

which converges absolutely for $|z| < 1$. However, it is more expedient to use the analytic continuation

$$ {}_2F_1(a,b;\,c;z) = (1-z)^{-a}\,{}_2F_1(a,c-b;\,c;\,z/(z-1)) \tag{5.9} $$

to construct a power series in terms of a new variable

$$ \xi_2 = \xi_2^*/(1+\xi_2^*) = \tau/t = 1-t_2^*/t_2 = 1-(t_1^*/t_1)^{\frac{3}{4}}, \tag{5.10} $$

which is less than unity for all $t_1 \geqslant t_1^* > 0$. With this transformation on (5.8), the asymptotic expansion (3.20) can be expressed as

$$ \begin{aligned} F(x;\,x^*) &= \left(\frac{2t_1}{Re}\right)^{\frac{1}{2}}\frac{Nu(x;\,x^*)}{\Psi_1^{(0)}(x)} \\ &= \frac{3}{\Gamma(\frac{1}{3})}\left(\frac{f_0''}{6}\right)^{\frac{1}{3}}\left(\frac{Pr}{\xi_2}\right)^{\frac{1}{3}}\left\{1-\frac{\Gamma(\frac{1}{3})}{180}\left(\frac{6}{f_0''}\right)^{\frac{4}{3}}\beta\,[{}_2F_1(\tfrac{4}{3},\tfrac{1}{3},\tfrac{8}{3};\xi_2)\right. \\ &\qquad \left. -\tfrac{1}{6}\xi_2\,{}_2F_1(\tfrac{4}{3},\tfrac{4}{3};\,\tfrac{11}{3};\,\xi_2)]\left(\frac{Pr}{\xi_2}\right)^{-\frac{1}{3}}+o(Pr^{-\frac{1}{3}})\right\}, \end{aligned} \tag{5.11} $$

where the function $F(x;\,x^*)$ is identical to that already defined by (4.38).

It should be noted here that, in the limits $t_1^* = 0$ or $t_1 \to \infty$, (5.11) reduces to the first two terms of the series given by Meksyn (1961) and by Merk (1959) for the case of a uniform surface temperature. This result can be deduced most readily by using the well-known relation

$$ {}_2F_1(a,b;c;\,1) = \Gamma(c)\,\Gamma(c-a-b)/\Gamma(c-a)\,\Gamma(c-b), \quad \text{for} \quad c \neq -1,\,-2,\dots,c > a+b. \tag{5.12} $$

It is of further interest to point out that the correction term in (5.11) to the asymptotic Nusselt number vanishes at the jump discontinuity in surface temperature, $t_1 = t_1^*$ ($x = x^*$) where $\xi_2 = 0$. This is a mathematical statement of the fact that Lighthill's formula for the asymptotic rate of heat transfer should become exact, even for small Pr, at the leading edge of a thermal boundary layer which originates downstream from the leading edge of the viscous layer, a result anticipated earlier, in § 4.2.

It is also of some value to inquire now as to the form of the series (5.11) in the case of (non-separating) flows with zero pressure gradient, which, in the planar case, corresponds to flow past a flat plate at zero incidence. Here one has $\beta = 0$ so that the correction term of $O(Pr^{-\frac{1}{3}})$ relative to the asymptotic Nusselt number vanishes from (5.11). However, in order to obtain the next higher-order correction term it suffices to take $p = 5$ in equation (3.1), with

$$ \overline{\psi}_5(x) \equiv \frac{1}{5!}\left(\frac{\partial^5\overline{\psi}}{\partial\overline{y}^5}\right)_{\overline{y}=0} = \frac{[\Psi_1^{(0)}(x)]^5}{(2t_1)^2}\frac{f_0^{\mathrm{v}}}{5!} = \frac{[\Psi_1^{(0)}(x)]^5}{(2t_1)^2}\left(\frac{2\beta-1}{120}\right)(f_0'')^2, $$

the latter equality being a consequence of (5.4). Thus, instead of (5.11), we obtain, on setting $\beta = 0$ in the preceeding relation,

$$F(x; x^*) = \frac{3}{\Gamma(\frac{1}{3})}\left(\frac{f_0''}{6}\right)^{\frac{1}{3}}\left(\frac{Pr}{\xi_2}\right)^{\frac{1}{3}}\left\{1 - \frac{1}{45\xi_2^2}[(2\xi_2 - 4)(1-\xi_2)\log(1-\xi_2) + \xi_2(5\xi_2 - 4)](Pr)^{-1} + o(Pr^{-1})\right\} \quad (5.13)$$

since the hypergeometric functions involved reduce here to the simple 'logarithmic' form. The remarks of the preceding paragraph are, of course, also applicable to the results of (5.13).

As a last application of the large-Pr expansions, we consider the case of a separating flow where $f_0'' = 0$ which, as is well known, occurs for

$$\beta = -0{\cdot}1988.... \quad (5.14)$$

In this instance, the formula (5.11) becomes singular due to the fact that the expression for the first term of (3.1) and, consequently, for the asymptotic Nusselt number, is no longer correct. However, the correct expansion can again be derived readily by the present method. Thus, since the first two non-vanishing derivatives of $\overline{\psi}$ at $\overline{y} = 0$ are now found to be the third and the seventh, we take $n = 3$ and $p = 7$ in (3.1), with $\overline{\psi}_3(x)$ given by the second equation of (5.5) and $\overline{\psi}_7$ by

$$\overline{\psi}_7(x) = \frac{[\Psi_1^{(0)}(x)]^7}{7!(2t_1)^3} f_0^{\text{vii}} = \frac{[\Psi_1^{(0)}(x)]^7}{(2t_1)^3}\left[\frac{2(3\beta-2)}{7!}\right]\beta^2,$$

the last equality resulting from (5.4). The appropriate form of the t-variable of (3.9) is now seen to be

$$t = \int_0^x [3\overline{\psi}_3(s)]^{\frac{1}{3}}\gamma_0(s)\,ds \equiv \tfrac{3}{4}(-\tfrac{1}{2}\beta)^{\frac{1}{3}}(2t_1)^{\frac{4}{3}}.$$

Therefore the functions R and A of (3.17) and (3.18) become

$$R(t) = 2(3\beta-2)\beta^2(2t_1)^{-\frac{4}{3}}/7!(-\beta/3)^{\frac{7}{3}} = 6(3\beta-2)t^{-1}/7!$$

and
$$A_3^{(7)}(\tau, t^*) = 6(3\beta-2)[\tfrac{7}{4}\tau^{\frac{3}{4}}(\tau+t^*)^{-1} - \tau^{\frac{7}{4}}(\tau+t^*)^{-2}]/7!.$$

Once again, when the preceding expression is substituted into (3.22) we encounter hypergeometric functions of the logarithmic type so that the final result for the Nusselt-number expansion becomes quite simply

$$F(x; x^*) = \frac{1}{\Gamma(\frac{1}{4})}\left(-\frac{32}{3}\beta\right)^{\frac{1}{4}}\left(\frac{Pr}{\xi_3}\right)^{\frac{1}{4}}\left\{1 + \frac{3\beta-2}{336\xi_3^2}[(8\xi_3 - 10)(1-\xi_3)\log(1-\xi_3) + (13\xi_3 - 10)\xi_3]Pr^{-1} + o(Pr^{-1})\right\}, \quad (5.15)$$

where
$$\xi_3 = 1 - (t_1^*/t_1)^{\frac{4}{3}} \quad (5.16)$$

and
$$\beta \equiv -0{\cdot}1988,....$$

Again, the remarks of the paragraph following equation (5.11) are also applicable to (5.15).

5.2. *Expansions for small Pr*

For similarity flows, the results for small Pr of §4, which are summarized by (4.38) and (4.39), can also be reduced to a much more elementary form. Since the variable t of §4, equation (4.25), will be identical henceforth to the variable t_1 of (5.2), we shall omit occasionally the subscript 1.

In the way of preliminary results, we note that the function $R(t)$ of (4.27) can be expressed, by means of (4.2), (5.3) and (5.4), as

$$R(t) = -\Delta_0(2t)^{\frac{1}{2}},$$

where
$$\Delta_0 = \Delta_0(\beta) = \lim_{\eta\to\infty}[\eta - f(\eta)] = \int_0^\infty [1-f'(\eta)]\,d\eta, \tag{5.17}$$

which is related to the 'displacement thickness' of the laminar boundary layer. Therefore, the corresponding function $A_1^{(0)}$ of (4.28) is merely

$$A_1^{(0)}(\tau, t^*) = -\Delta_0\{2(\tau+t^*)\}^{-\frac{1}{2}}. \tag{5.18}$$

Similarly, we find, for the function $M(t)$ of (4.33),

$$M(t) = 2\Delta_1 t \tag{5.19}$$

where
$$\Delta_1 = \Delta_1(\beta) = \int_0^\infty \eta[1-f'(\eta)]\,d\eta = \int_0^\infty [f(\eta)-(\eta-\Delta_0)]\,d\eta.$$

The result given by (4.39) can now be simplified considerably by means of (5.18) and (5.19). Thus, letting

$$\xi_1 = \tau/t \equiv 1 - t_1^*/t_1, \tag{5.20}$$

we obtain for the first function in (4.39)

$$K_0^{(0)}(t;\, t^*) = (2/\pi\xi_1)^{\frac{1}{2}}. \tag{5.21}$$

On the other hand, because of (4.30) and (5.18), together with (5.8) and (5.9), we find that

$$-\left(\frac{\partial\theta_0^{(1)}}{\partial z}\right)_{z=0} = -\frac{\Delta_0}{(2\pi^2)^{\frac{1}{2}}}\int_0^1 \left(\frac{\lambda}{1-\lambda}\right)^{\frac{1}{2}}(\lambda\tau+t^*)^{-\frac{1}{2}}\,d\lambda$$
$$= -[\Delta_0/2(2t)^{\frac{1}{2}}]\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};\,2;\xi_1),$$

whence it follows that

$$K_0^{(1)}(t;\, t^*) = [2\delta(\xi_1) - \tfrac{1}{2}\Delta_0\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};\,2;\,\xi_1)], \tag{5.22}$$

which is clearly a function only of ξ_1. Furthermore, on account of (5.12),

$$K_0^{(1)}(t;\, 0) \equiv K_0^{(1)}(\infty;\, t^*) = -2\Delta_0/\pi. \tag{5.23}$$

The evaluation of the third function of (4.39) poses more of a problem, however. In the first place the use of equations (4.36), (4.37) and (5.19) results in the following rather complicated expression

$$K_0^{(2)}(t;\, t^*) = \left(\frac{2}{\pi\xi_1}\right)^{\frac{1}{2}}\left\{2\Delta_1\left[2\xi_1^{\frac{1}{2}}\frac{d}{d\xi_1}\left(\frac{H(\xi_1)}{\xi_1^{\frac{1}{2}}}\right)+\tfrac{3}{2}\right]\right.$$
$$\left.+\frac{\Delta_0^2}{2\pi}\,\xi_1^*\int_0^1\int_0^1[(\lambda'\xi_1^*+1)(\lambda\lambda'\xi_1^*+1)]^{-\frac{1}{2}}f(\lambda,\lambda')\,d\lambda'\,d\lambda\right\},$$

where H is the Heaviside function, a, b, c and f are the functions of λ and λ' defined in (4.36), and

$$\xi_1^* = t/t^* - 1 = \xi_1/(1-\xi_1),$$

ξ_1 being the variable of (5.20). Fortunately, the double integral appearing here can be developed in a power series in ξ_1^*, which can be expressed subsequently in terms of known functions. To demonstrate this, we first let

$$\rho = \lambda'\lambda^{\frac{1}{2}}\xi_1^*, \quad s = \tfrac{1}{2}(\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}),$$

so that the term containing ξ_1^* in the above integral becomes

$$(\lambda'\xi_1^* + 1)(\lambda'\lambda\xi_1^* + 1) = \rho^2 + 2s\rho + 1.$$

Then, by substituting into the integrand the well-known expansion in terms of the Legendre polynomials P_n

$$(\rho^2 + 2s\rho + 1)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} (-\rho)^n P_n(s),$$

we obtain, on termwise integration,

$$K_0^{(2)}(t;\, t^*) = \left(\frac{2}{\pi\xi_1}\right)^{\frac{1}{2}} \left\{2\Delta_1 \left[2\xi_1^{\frac{1}{2}} \frac{d}{d\xi_1}\left(\frac{H(\xi_1)}{\xi_1^{\frac{1}{2}}}\right) + \frac{3}{2}\right] + \frac{\Delta_0^2 \xi_1^*}{2\pi} \sum_{n=0}^{\infty} C_n(-\xi_1^*)^n\right\}, \tag{5.24}$$

where $$C_n = \int_0^1\int_0^1 (\lambda')^n (\lambda)^{n/2} P_n(\tfrac{1}{2}\lambda^{\frac{1}{2}} + \tfrac{1}{2}\lambda^{-\frac{1}{2}}) f(\lambda, \lambda')\, d\lambda'\, d\lambda$$

for $n = 0, 1, 2, \ldots$. Since the above Legendre expansion is absolutely convergent for $|\xi_1^*| < 1$, in the region $0 \leqslant \lambda, \lambda' \leqslant 1$, it follows that the power series in (3.24) will be absolutely convergent for $|\xi_1^*| < 1$, and therefore, for $t^* \leqslant t < 2t^*$. Moreover, we shall show in Appendix 2 that this series can be expressed in terms of products of Legendre functions of the type

$$Q_\nu^\mu(z) = 2^\nu e^{i\mu\pi} \frac{\Gamma(1+\nu)\,\Gamma(1+\nu+\mu)}{\Gamma(2+2\mu)} \frac{(z+1)^{\frac{1}{2}\mu-\nu-1}}{(z-1)^{\frac{1}{2}\mu}}\, {}_2F_1(1+\nu-\mu, 1+\nu; 2+2\nu; 2/(1+z)) \tag{5.25}$$

to give, finally, for (5.24),

$$K_0^{(2)}(t;\, t^*) = \left(\frac{2}{\pi\xi_1}\right)^{\frac{1}{2}} \left\{2\Delta_1 \left[2\xi_1^{\frac{1}{2}} \frac{d}{d\xi_1}\left(\frac{H(\xi_1)}{\xi_1^{\frac{1}{2}}}\right) + \frac{3}{2}\right] + \frac{\Delta_0^2}{2\pi} N(\xi_1)\right]\right\}, \tag{5.26}$$

where $$N(\xi_1) = [(1-\xi_1)/\xi_1]\,[Q_{-\frac{1}{4}}Q_{\frac{1}{4}} + (1-\xi_1)^{-\frac{1}{2}}(Q_{\frac{1}{4}}Q^1_{-\frac{1}{4}} - Q_{-\frac{1}{4}}Q^1_{\frac{1}{4}})],$$

with $$Q_\nu^\mu \overset{\text{def}}{=} Q_\nu^\mu(2\xi_1^{-1} - 1), \quad Q_\nu \overset{\text{def}}{=} Q_\nu^0.$$

The limiting form of (5.26) for $t^* \to 0$ or $t \to \infty$ ($\xi_1 \to 1$) can be deduced readily from the relations (Erdelyi *et al.* 1954 H.T.F., vol. 1)

$$\left.\begin{aligned} &Q_\nu^1(z) \to 2^{-\frac{1}{2}}(z-1)^{\frac{1}{2}} \\ \text{and}\quad &Q_\nu(z) \to -\tfrac{1}{2}\log(\tfrac{1}{2}z - \tfrac{1}{2}) - \dot{\gamma} - \psi(\nu+1) \end{aligned}\right\} \quad \text{for} \quad z \to 1,$$

where $\dot{\gamma}$ is Euler's constant, and ψ denotes here the logarithmic derivative of the gamma function. Thus, we find that

$$\begin{aligned} K_0^{(2)}(t;\, 0) \equiv K_0^{(2)}(\infty;\, t^*) &= (2/\pi)^{\frac{1}{2}}\{\Delta_1 + (\Delta_0^2/2\pi)\,[\psi(\tfrac{5}{4}) - \psi(\tfrac{3}{4})]\} \\ &\equiv (2/\pi)^{\frac{1}{2}}[\Delta_1 + (\Delta_0^2/2\pi)(4-\pi)]. \end{aligned} \tag{5.27}$$

In order now to summarize formally the results of this section we note that for a given differentiable surface temperature distribution $\theta_s(x) \equiv h(t)$ we can apply the superposition formula of (4.40) together with (5.21), (5.22) and (5.26) to derive the following asymptotic relation for heat transfer to 'similar' boundary layer flows:

$$\begin{aligned}
-\frac{1}{\Psi_1^{(0)}(x)}\left(\frac{\pi t}{Pe}\right)^{\frac{1}{2}}\left(\frac{\partial\theta}{\partial y}\right)_{y=0} \\
\equiv -\frac{1}{\gamma_0(x)}\left(\frac{\pi t^{1-\beta}}{Pe}\right)^{\frac{1}{2}}\left(\frac{\partial\theta}{\partial y}\right)_{y=0} \\
= h(0)+t\int_0^1 \frac{h'\{t(1-\xi_1)\}}{\xi_1^{\frac{1}{2}}}d\xi_1 \\
-\Delta_0\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left\{\frac{2h(0)}{\pi}+t\left[\frac{1}{2}\int_0^1 {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};\,2;\,\xi)\,h'\{t(1-\xi)\}\,d\xi-2h'(t)\right]\right\}(Pr)^{\frac{1}{2}} \\
+\left\{\left(\Delta_1+\frac{4-\pi}{4\pi}\Delta_0^2\right)h(0)+\Delta_1 t\left[2h'(0)+3\int_0^1\frac{h'\{t(1-\xi)\}}{\xi^{\frac{1}{2}}}d\xi\right.\right. \\
\left.\left.+4t\int_0^1\frac{h''\{t(1-\xi)\}}{\xi^{\frac{1}{2}}}d\xi+\frac{\Delta_0^2 t}{2\pi}\int_0^1 N(\xi)\,h'\{t(1-\xi)\}\,d\xi\right\}Pr+o(Pr)+O(Pe^{-\frac{1}{2}}),
\end{aligned} \tag{5.28}$$

where the primes denote differentiation with respect to t ($\equiv t_1$) and $N(\xi)$ is the function appearing in (5.26). The appropriate expression for the term of $O(Pe^{-\frac{1}{2}})$ (zero for planar flows) could also have been inserted from the results in Appendix 1.

As a specific example of the application of (5.28), we consider now the case of a planar 'wedge-type' flow with a surface temperature which increases as a power of the distance along the surface. In our present notation this can be stated as

$$\gamma_0(x) \equiv 1, \quad U(x) = x^m/(m+1)^{\frac{1}{2}\beta} = t_1^{\frac{1}{2}\beta},$$
$$\theta_s(x) \equiv h(t_1) = x^s = (m+1)^{\nu}\,t_1^{\nu},$$

where

$$\beta = \frac{2m}{m+1}, \quad \nu = \frac{s}{m+1} > 0.$$

In this case, the first two integrals occurring in (5.28) become, respectively,

$$\nu\int_0^1 \xi^{-\frac{1}{2}}(1-\xi)^{\nu-1}d\xi = \frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}$$

and

$$\nu\int_0^1 {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};\,2;\,\xi)\,(1-\xi)^{\nu-1}d\xi \equiv 4\nu\,{}_2F_1(-\tfrac{1}{2},-\tfrac{1}{2};\,\nu;\,1) = 4[\Gamma(\nu+1)/\Gamma(\nu+\tfrac{1}{2})]^2,$$

the first result being the well-known expression for the 'beta' integral, and the second following on termwise integration of the power series for ${}_2F_1$ with subsequent application of (5.12). Thus, we find for (5.28) that

$$\begin{aligned}
-\frac{1}{[PexU(x)]^{\frac{1}{2}}}\frac{x}{\theta_s(x)}\left(\frac{\partial\theta}{\partial y}\right)_{y=0} \\
= (m+1)^{1/(2m+2)}\left\{\frac{\Gamma(1+\nu)}{\Gamma(\frac{1}{2}+\nu)}-\sqrt{2}\Delta_0\left[\left(\frac{\Gamma(1+\nu)}{\Gamma(\frac{1}{2}+\nu)}\right)^2-\nu\right](Pr)^{\frac{1}{2}}+O(Pr)+O(Pe^{-\frac{1}{2}})\right\},
\end{aligned} \tag{5.29}$$

to the order of terms indicated. Since ν is arbitrary, (5.29) represents an extension of the result of Morgan *et al.* (1958) which applies only as long as ν is an integer.

In order to specify the term of $O(Pr)$ in (5.29) it is necessary to evaluate integrals of the form
$$\int_0^1 (1-\zeta)^\nu \zeta^{-\frac{3}{2}} Q_{\frac{1}{4}}^{\mu}\left(\frac{2-\zeta}{\zeta}\right) Q_{-\frac{1}{4}}^{\lambda}\left(\frac{2-\zeta}{\zeta}\right) d\zeta$$
(where λ and μ assume independently the values zero or unity) which we were unable to express in terms of known functions. However, for the special case of a uniform surface temperature, say
$$\theta_s(x) = h(t) = 1,$$
the term in question is given immediately by (5.27). Thus, taking account of (5.23), we find that (5.28) reduces in this case to
$$-\frac{1}{[Pe\, U(x)]^{\frac{1}{2}}}\left(\frac{\partial\theta}{\partial y}\right)_{y=0}$$
$$= \left(\frac{(m+1)^{1/(m+1)}}{\pi x}\right)^{\frac{1}{2}}\left\{1-\sqrt{\frac{2}{\pi}}\Delta_0(Pr)^{\frac{1}{2}}+\left[\left(\frac{4-\pi}{2\pi}\right)\Delta_0^2+\Delta_1\right]Pr+O(Pr^{\frac{3}{2}})+O(Pe^{-\frac{1}{2}})\right\}. \tag{5.30}$$
We recall that the constants Δ_0 and Δ_1, which depend on the 'wedge parameter' β, are defined by (5.17) and (5.19); these could then be evaluated from the tabulated numerical solutions of (5.4). Considering, for example, the special case of a flat plate at zero incidence, where $m = \beta = 0$, $U(x) \equiv 1$, we have (from Morgan *et al.* 1958) that $\Delta_0 = 1{\cdot}2167, \ldots$. As for Δ_1, we can employ the differential equation of (5.14) to express the integral of (5.19) as
$$\Delta_1 = \int_0^\infty \left[\frac{f'''}{f''}+(\eta-\Delta_0)\right]d\eta = -\log\{f''\, e^{\frac{1}{2}(\eta-\Delta_0)^2}\}\Big|_0^\infty = \tfrac{1}{2}\Delta_0^2+\log\{f''(0)/c\},$$
where
$$c = \lim_{\eta\to\infty} f''(\eta)\, e^{\frac{1}{2}(\eta-\Delta_0)^2}.$$
Now, with the well-known numerical value
$$f''(0) = (0{\cdot}33206\ldots)\sqrt{2}$$
and with c taken from Blasius's (Schlichting 1955) asymptotic expression for $f(\eta)$,
$$c \doteq 0{\cdot}231\sqrt{2},$$
we have
$$\log\{f''(0)/c\} \doteq \log(0{\cdot}332/0{\cdot}231) = 0{\cdot}3626\ldots.$$
Thus, (5.30) becomes
$$-(x/Re)^{\frac{1}{2}}(\partial\theta/\partial y)_{y=0} = 0{\cdot}564Pr^{\frac{1}{2}} - 0{\cdot}548Pr + 0{\cdot}508(Pr)^{\frac{3}{2}} + o(Pr^{\frac{3}{2}})$$
to terms of $O(Pe^{-1})$.

In closing, we should point out that in this special case the expansion could have been derived directly from the well-known integrals for heat transfer with 'similar' velocity and temperature distributions, by using a method analogous to that described by Merk (1959).

As a last remark, we note that the term of $O(Pr^{\frac{3}{2}})$, which we have derived here, would tend to indicate a rather poor convergence of the small Pr expansions, a fact noted also by Morgan *et al.* (1958). For example, at $Pr = 0{\cdot}1$ the magnitude of this term is already about 10 % of the leading term. On this basis, and in light of the remarks of § 4, one might speculate that, in general, the small Pr expansions represent semi-convergent series, whose 'apparent' convergence is further weakened by rapidly varying surface temperatures. By contrast, the large-Pr expansion would appear to be a convergent series (in descending powers of Pr) whose convergence becomes extremely rapid near points where the surface temperature is changing rapidly. However, since the 'boundary-layer' approximation (equation (2.3)) for the temperature field is weakened precisely at such points, this does not, of course, imply that the large-Pr expansions give an adequate description of forced convection near points of abrupt surface-temperature changes.

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Appendix 1. Derivation of the surface curvature correction in the expansion for small *Pr*

The following is a brief derivation of the $O(Pe^{-\frac{1}{2}})$ term in the small Pr expansion of (4.5), as well as the corresponding $O(Pr^{\frac{1}{2}}\,Pe^{-\frac{1}{2}})$ term in the expansion (4.10). These terms yield, in turn, the $O(Pr^{\frac{1}{2}}Pe^{-\frac{1}{2}})$ correction in (4.38).

By employing the expansion technique discussed by Acrivos & Goddard (1965) we can show that the function $\theta_1^{(0)}$ of (4.5) must satisfy the partial differential equation

$$P_0^{(0)}\theta_1^{(0)} = -P_1^{(0)}\theta_0^{(0)}, \tag{A 1.1}$$

where $P_0^{(0)}$ is the first differential operator of (4.8), $\theta_0^{(0)}$ is the first term of (4.5), taken here to be the function in (4.23), and $P_1^{(0)}$ denotes the differential operator

$$P_1^{(0)} = 2\Psi_2^{(0)}y_1\frac{\partial}{\partial x} - \Psi_2^{(0)\prime}y_1^2\frac{\partial}{\partial y_1} - (\gamma_1+\gamma_0\alpha_1)\frac{\partial}{\partial y_1}y_1\frac{\partial}{\partial y_1}. \tag{A 1.2}$$

Here $\Psi_2^{(0)}$ is the second functional coefficient in the stream-function expansion (4.3), the function $\gamma_0(x)$ has the same significance as in the main text above, i.e. it is identically equal to unity for planar flows and equal to the dimensionless radius of revolution of the heat-transfer surface for axisymmetric flows, and the function $\alpha_1(x)$ is the dimensionless surface curvature in the plane of flow. Moreover, in the case of planar flow $\gamma_1(x) \equiv 0$, and for axisymmetric flow γ_1 and α_1 are related to γ_0 by

$$\gamma_1 = [1-(\gamma_0')^2]^{\frac{1}{2}}, \quad \alpha_1 = -\gamma_0''/\gamma_1', \tag{A 1.3}$$

the primes denoting differentiation with respect to x.

Since the function $\Psi^{(0)}$ of (4.1) and (4.3) is assumed to be the stream function for a potential flow, we can show that the function $\Psi_2^{(0)}(x)$ of (4.3), which appears above in (A 1.2), must always be related to the first coefficient in (4.3) by

$$\Psi_2^{(0)}(x) = \tfrac{1}{2}\Psi_1^{(0)}(x)\,[\gamma_1/\gamma_0-\alpha_1]. \tag{A 1.4}$$

Now, since $\theta_1^{(0)}$ is to satisfy (A 1.1) and (4.9) and to vanish at $y_1 = 0$ because of (4.14), it must be given by the integral Λ of (3.11) with

$$q \equiv -P_1^{(0)}\theta_0^{(0)}, \tag{A 1.5}$$

$P_1^{(0)}$ being the operator defined by (A 1.2). Moreover, we find that this operator can be expressed in terms of the variables (z, t) of (4.25) as

$$P_1^{(0)} = J^{-1}\left\{-\left(\frac{\gamma_1}{\gamma_0}+\alpha_1\right)\frac{1}{\Psi_1'^{(0)}}\frac{\partial}{\partial z}z\frac{\partial}{\partial z}+\left[2R(t)\,z\frac{\partial}{\partial t}-R'(t)\,z^2\frac{\partial}{\partial z}\right]\right\}, \tag{A 1.6}$$

where
$$R(t) = \frac{\Psi_2'^{(0)}}{[\Psi_1'^{(0)}]^2} = \frac{1}{2\Psi_1'^{(0)}}\left[\frac{\gamma_1}{\gamma_0}-\alpha_1\right],$$

the latter equality resulting from (A 1.4). Noting further that the last two terms in (A 1.6) are a special case of the operator of (3.17) and, as before, that $\theta_0^{(0)}$, as given by (4.23), is a special case of the function (3.16), we can deduce that the contribution of these terms to the function q of (A 1.5) will be necessarily a special case of (3.18). Finally, we can show readily by (4.23) and (4.25) that

$$-\frac{\partial}{\partial z}z\frac{\partial}{\partial z}\theta_0^{(0)} = \frac{1}{\Gamma(\frac{1}{2})\,\tau^{\frac{1}{2}}}\left[1-\frac{(2\zeta)^2}{2}\right]e^{-\zeta^2}.$$

In other words, the first term in (A 1.6) represents a contribution to (A 1.5) which again is a linear combination of terms like those of (3.18). It follows from these considerations that the function q of (A 1.5) will be essentially a linear combination of terms like those of (3.18), and that, therefore, $\theta_1^{(0)}$ is expressible as a linear combination of functions like those of (3.19). In particular, we can show in a straightforward way that

$$\theta_1^{(0)}(t,z;\,t^*) = \frac{1}{\pi}ze^{-\zeta^2}\int_0^1\exp\left(-\frac{\lambda}{1-\lambda}\zeta^2\right)\lambda^{\frac{1}{2}}\left[4(1-\lambda)^{\frac{1}{2}}A_1^{(2)}(\lambda\tau^*,t)\right.$$
$$\left.{}_1F_1\left(2;\,\tfrac{3}{2};\,\frac{\zeta^2\lambda}{1-\lambda}\right)-(1-\lambda)^{-\frac{1}{2}}A_1^{(0)}(\lambda\tau,t^*)\,{}_1F_1\left(1;\,\tfrac{3}{2};\,\frac{\zeta^2\lambda}{1-\lambda}\right)\right]d\lambda, \tag{A 1.7}$$

where
$$A_1^{(2)}(\tau,t^*) = \frac{1}{\tau}\frac{\partial}{\partial\tau}[\tau^2R^{(1)}(\tau+t^*)]-\tau\frac{\partial}{\partial\tau}R^{(0)}(\tau+t^*),$$
$$A_1^{(0)}(\tau,t^*) = 2[R^{(0)}(\tau+t^*)+R^{(1)}(\tau+t^*)],$$
with
$$R^{(1)}(t) = \alpha_1/2\Psi_1'^{(0)},\quad R^{(0)} = \gamma_1/2\gamma_0\Psi_1'^{(0)}.$$

As it turns out, the resulting expression for the surface derivative of $\theta_1^{(0)}$ can be reduced simply to

$$\left(\frac{\partial\theta_1^{(0)}}{\partial z}\right)_{z=0} = \frac{4}{\pi}\int_0^1(1-2\lambda)\left(\frac{\lambda}{1-\lambda}\right)^{\frac{1}{2}}R^{(0)}(\lambda\tau+t^*)\,d\lambda, \tag{A 1.8}$$

which involves $R^{(0)}$ only.

Now, by generalizing the expansion of Morgan *et al.* (1958), one can show that the corresponding function $\bar{\theta}_1^{(0)}$ in the inner expansion (4.10) must satisfy the differential equation

$$\frac{\partial^2\bar{\theta}_1^{(0)}}{\partial\bar{y}^2} = -(\gamma_1+\gamma_0\alpha_1)\frac{\partial}{\partial\bar{y}}\bar{y}\frac{\partial}{\partial\bar{y}}\bar{\theta}_0^{(0)} \equiv 0,$$

the last equality arising from (4.15). Hence, it follows that $\bar{\theta}_1^{(0)}$ is a linear homogeneous function of $\bar{y}$, and by continuing the matching procedure of §4.3, we can show that it is indeed the function indicated by (4.22).

Finally, we see that equation (A 1.8) gives the desired correction term of $O(Pr^{\frac{1}{2}}Pe^{-\frac{1}{2}})$ in (4.38). For planar flows where $\gamma_0 \equiv 1$, this term vanishes identically, which allows us to conclude that this correction term is due solely to surface curvature in planes normal to the plane of flow. In fact, for planar flows, the function of (A 1.7) reduces to a form which can be derived from Boussinesq's (1903) classical formula for the temperature distribution in a potential flow field since, due to his co-ordinate transformation which employs the stream and potential functions as independent variables, Boussinesq's expression for the temperature distribution is correct to terms of $O(Pe^{-1})$ for $Pr \to 0$, provided that the surface temperature is continuous, whereas the function $\theta_1^{(0)}$ of (4.23) is, of course, correct only to terms of $O(Pe^{-\frac{1}{2}})$. However, the expression for heat transfer given by (4.23) is identical to Boussinesq's result, as shown by the fact that the derivative (A 1.8) vanishes when $\gamma_1 \equiv 0$.

Appendix 2. Transformation of a power series in the small *Pr* expansion

We wish to show here now the power series in (5.24) can be continued analytically to yield (5.26). Although admittedly somewhat tedious, the derivation involves the application of some of the rather remarkable relations between hypergeometric functions which should make it of sufficiently general interest to warrant its outline here. Since we shall make frequent use of relations given by Erdelyi *et al.* (1945 H.T.F., vol. 1) we shall refer to this work simply as E.

First of all, it is a relatively easy matter to show that the double integral of (5.24) for the constant C_n can be reduced to a single integral involving hypergeometric functions. In particular, using the definition of a, b, c, and f in (4.36) together with the series

$$\tan^{-1}s = \sum_{k=0}^{\infty} \frac{(-1)^k s^{2k+1}}{2k+1} \quad (|s| < 1)$$

we obtain by a termwise integration with respect to λ' and by an application of the relations (5.7) and (5.9), that

$$C_n = \frac{\sqrt{\pi}(n+1)!}{\Gamma(n+\frac{1}{2})} \int_0^1 \lambda^{\frac{1}{2}n+1}(1-\lambda)^{-\frac{1}{2}} P_n(\tfrac{1}{2}\lambda^{\frac{1}{2}} + \tfrac{1}{2}\lambda^{-\frac{1}{2}}) \left\{ {}_2F_1\left(\tfrac{1}{2}, 1;\, n+\tfrac{5}{2};\, \frac{\lambda}{1-\lambda}\right) \right.$$
$$\left. - \frac{1}{2(n+\frac{5}{2})}\left[{}_2F_1\left(\tfrac{1}{2}, 1;\, n+\tfrac{7}{2};\, \frac{\lambda}{1-\lambda}\right) + {}_2F_1\left(\tfrac{3}{2}, 1;\, n+\tfrac{7}{2};\, \frac{\lambda}{1-\lambda}\right)\right]\right\} d\lambda. \quad \text{(A 2.1)}$$

However, by taking $a = 1$, $b = \frac{1}{2}$, $c = n+\frac{1}{2}$ in one of the relations of Gauss (E, p. 103):

$$(c-1)\,{}_2F_1(a,b;\, c-1;\, z) - b\,{}_2F_1(a,b+1;\, c;\, z) = (c-b-1)\,{}_2F_1(a,b;\, c;\, z)$$

for 'contiguous' hypergeometric functions, we can reduce the linear combination of functions in the integrand to a single function. By employing subsequently the transformation (5.9) on this function we find that

$$C_n = \frac{\sqrt{\pi}\,(n+1)!\,(n+\frac{3}{2})}{\Gamma(n+\frac{7}{2})} \int_0^1 \lambda^{\frac{1}{2}n+\frac{1}{2}} P_n(\tfrac{1}{2}\lambda^{\frac{1}{2}} + \tfrac{1}{2}\lambda^{-\frac{1}{2}})\, {}_2F_1(\tfrac{1}{2}, n+\tfrac{5}{2};\, n+\tfrac{7}{2};\, \lambda)\, d\lambda.$$

Next, by replacing the hypergeometric function here by its power series in λ (which is convergent for $0 \leqslant \lambda \leqslant 1$) and by integrating termwise with the aid of the relations (Erdelyi *et al.* 1954, T.I.T., vol. 1, p. 171)

$$\int_0^1 \lambda^\nu P_n(\tfrac{1}{2}\lambda^{\frac{1}{2}}+\tfrac{1}{2}\lambda^{-\frac{1}{2}})\,d\lambda = 2\int_0^\infty e^{-2(\nu+1)\theta}P_n(\cosh\theta)\,d\theta$$
$$= \frac{\Gamma(\nu+\frac{1}{2}n+\frac{3}{2})\,\Gamma(\nu-\frac{1}{2}n+1)}{\Gamma(\nu-\frac{1}{2}n+\frac{3}{2})\,\Gamma(\nu+\frac{1}{2}n+2)} \quad (\nu > \tfrac{1}{2}n-1)$$

we obtain

$$C_n = \frac{\pi}{2}\frac{[(n+1)\,!\,]^2(n+\frac{3}{2})}{\Gamma(n+\frac{7}{2})\,\Gamma(n+\frac{5}{2})}\,{}_3F_2(\tfrac{1}{2}, n+2, \tfrac{3}{2};\; n+\tfrac{7}{2}, 2;\; 1),$$

where ${}_3F_2$ denotes the generalized hypergeometric series, here of argument unity. However, since one of its 'upper' parameters $(n+2)$ exceeds one of the 'lower' parameters (2) by an integral value n, one can, by n-fold application of the elementary relation (E, pp. 188–190)

$${}_3F_2(a+1,b,c;\; d,e;\; 1) = {}_3F_2(a-1,b,c;\; d,e;\; 1)$$
$$+\,(bc/de)\,{}_3F_2(a,b+1,c+1;\; d+1,e+1;\; 1),$$

reduce the hypergeometric function in the preceding relation to a finite sum of simpler hypergeometric functions ${}_2F_1$ with unit argument, and with the aid of (5.12) this sum can in turn be expressed in terms of gamma functions. In this manner, we find that

$$C_n = \tfrac{1}{2}\pi\{(n+\tfrac{3}{2})/(n+2)\}\,{}_3F_2(\tfrac{1}{2},\tfrac{3}{2},-n;\; -n-\tfrac{1}{2},2;\; 1), \tag{A 2.2}$$

where the hypergeometric function ${}_3F_2$, directly expressible in terms of the above sum, has a terminating series.

In order now to sum the series of (5.24) we shall first make use of the fact that

$$(1-z)^{-d}\,{}_2F_1\left(a,b;\; c;\; \frac{z}{z-1}\right) = \sum_{k=0}^{\infty}\frac{\Gamma(d+k)}{\Gamma(d)\,k\,!}\;{}_3F_2(-k,a,b;c,d;1)\,z^k \tag{A 2.3}$$

whenever the series involved are convergent (E, p. 187). Applying next the transformation (5.9) to the hypergeometric function on the left-hand side of (A 2.3) and in turn applying (A 2.3) to the transformed function, we can deduce by comparing like powers of z that

$${}_3F_2(-k,a,b;c,a-d+1-k;\; 1) = [\Gamma(d-a)\,\Gamma(d+k)/\Gamma(d-a-k)\,\Gamma(d)]$$
$$\times\,{}_3F_2(-k,a,b-c;\; c,d;\; 1)$$

and that

$${}_3F_2(-k,a,b;\; c,d;\; 1) = [\Gamma(d+c-a-b+k)\,\Gamma(d)/\Gamma(d+k)\,\Gamma(d+c-a-b)]$$
$$\times\,{}_3F_2(-k,c-a,c-b;\; d+c-a-b,c;\; 1),$$

which together with the preceding equation for C_n gives

$$C_n = \tfrac{1}{2}\pi[(n+1)\,!\,\Gamma(\tfrac{3}{2})/(n+2)\,\Gamma(n+\tfrac{3}{2})]\,{}_3F_2(-n,\tfrac{1}{2},\tfrac{1}{2};\; 2,2;\; 1)$$
$$= \tfrac{1}{4}\pi[(n+1)\,!\,\Gamma(\tfrac{3}{2})/\Gamma(n+\tfrac{3}{2})]\,{}_3F_2(-n,\tfrac{3}{2},\tfrac{3}{2};\; 3,2;\; 1). \tag{A 2.4}$$

Next, by inspecting the latter expression for C_n and (A 2.3) (with $a = b = \frac{3}{2}$, $c = 3$ and $d = 2$) we see that

$$\sum_{n=0}^{\infty} A_n z^n = (1-z)^{-2}\,{}_2F_1(\tfrac{3}{2},\tfrac{3}{2};\ 3;\ z/z-1) = (1-z)^{-\frac{1}{2}}\,{}_2F_1(\tfrac{3}{2},\tfrac{3}{2};\ 3;\ z),$$

if we take A_n to be

$$A_n = \frac{4}{\pi}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(\frac{3}{2})\,n\,!}C_n = (n+1)\,{}_3F_2(-n,\tfrac{3}{2},\tfrac{3}{2};\ 3,2;\ 1).$$

However, from a theorem of Cayley and Orr (E, pp. 82–83) which states, that, if

$$(1-z)^{a+b-c}\,{}_2F_1(2a,2b;\ 2c-1;\ z) = \sum_{n=0}^{\infty} A_n z^n,$$

then $${}_2F_1(a,b;\ c-\tfrac{1}{2};\ z)\,{}_2F_1(c-a,c-b;\ c+\tfrac{1}{2};\ z) = \sum_{n=0}^{\infty}\frac{\Gamma(c+n)\,\Gamma(c+\frac{1}{2})}{\Gamma(c)\,\Gamma(c+n+\frac{1}{2})}A_n z^n,$$

it follows (by taking $a = b = \frac{3}{4}$, $c = 2$) that, with the aid of (A 2.4),

$$\begin{aligned} F(z) &\overset{\text{def}}{=} \sum_{n=0}^{\infty}\frac{(n+1)\,!\,\Gamma(\frac{5}{2})}{\Gamma(n+\frac{5}{2})}A_n z^n \\ &= \frac{6}{\pi}\sum_{n=0}^{\infty}\frac{(n+1)}{(n+\frac{3}{2})}C_n z^n = {}_2F_1(\tfrac{3}{4},\tfrac{3}{4};\ \tfrac{3}{2};\ z)\,{}_2F_1(\tfrac{5}{4},\tfrac{5}{4};\ \tfrac{5}{2};\ z). \end{aligned} \tag{A 2.5}$$

Finally, with this result, we can easily show that

$$\sum_{n=0}^{\infty} C_n z^n = \frac{\pi}{6}\left\{F(z) + \frac{1}{2z}\int_0^z F(z')\,dz'\right\} \tag{A 2.6}$$

whenever the various series converge. Now, by (5.25), the hypergeometric functions on the right-hand side of (A 2.5) can be expressed as a product of Legendre functions, so that one can employ the indefinite integral (E, p. 170),

$$(\gamma-\nu)(\nu+\gamma+1)\int^z Q_\nu(z')\,Q_\gamma(z')\,dz' = (z^2-1)^{\frac{1}{2}}\,[Q_\gamma(z)\,Q_\nu^1(z) - Q(z)\,Q_\nu^1(z)]$$

to expess the right-hand side of (A 2.6) as a product of such functions. Equation (5.26) follows then from (5.24) in a straightforward manner.

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